Technical Mechanics

For

Engineering Students

By

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With 343 Diagrams and
155 Illustrative Examples

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Dedicated

To

my revered teacher

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अज्ञानतिमिरान्धस्य ज्ञानाञ्जनशलाकया ।
चक्षुरुन्मीलितं येन तस्मै श्रीगुरवे नमः ॥

PREFACE

I had long been aware of the difficulties that face our students on account of the want of a suitable textbook on Technical Mechanics and the urgent need to remove them. The cordial reception that was accorded to my first book—"Mechanics—Part I" (Dynamics of Paritcles)—by the students and teachers of our College has encouraged me to undertake the writing of a complete volume on Elementary Technical Mechanics. Although the manuscript of this book was ready for the press by June, 1948, the actual printing for various reasons, was delayed and the book has had to be rushed through the press within a very short period. As such, it is very likely that some printing mistakes may have crept in. I shall be greatly obliged if readers will kindly bring to my notice any such errors.

In writing this book particular care and attention has been taken in the selection of a large number of illustrative examples prepared specially for this work. The methods of treatment that have been adopted in the solution of these examples will, I hope, enable the students, after they have attended class lectures, to tackle successfully with various types of problems set in different textbooks.

Graphical Treatment is another special feature of this volume. As graphical method is often found to be very useful in the solution of problems, an exhaustive treatment has been made of this method in this volume.

I shall feel amply rewarded if the book proves useful and beneficial to those for whom it has been intended.

I avail myself of this opportunity to acknowledge my indebtedness to Prof. G. C. Sen, M.S. (Michigan) and Prof. H. G. Ganguli of our College whose valuable suggestions have been of great assistance to me.

I am also extremely grateful to Dr. T. Sen, Principal of our College, for the generous encouragement I have always received from him.

Lastly, I shall be failing in my duty if I fail to record my deep sense of gratitude to Sri Santi Comar Ghose, Managing Director, Calcutta Press Ltd., but for whose untiring energy this book would not have seen the light of day in such a remarkably short period of time.

Calcutta,
June, 1949.

AMIYAKUMAR BASU

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CHAPTER 1

INTRODUCTORY

1. **Division of Subject.** The term 'Mechanics' was used by Newton to mean the science of machines and the art of making them. But gradually the idea changed and it is now being used to represent the science of motion and force. This science is generally divided into three parts.

I. Kinematics—the science of motion without any reference to the mass moved and the cause of motion in it.

II. Kinetics—the science that deals with the unbalanced forces, that is to say, the science that establishes the relation between motion and force with respect to the mass moved.

III. Statics—the science of the balanced forces, *i.e.*, the forces that act on a mass without creating any motion in it.

2. **Mechanics can again be divided into three divisions as follows :**

I. Mechanics of Particles.

II. Mechanics of Rigid Bodies.

III. Mechanics of Elastic Bodies.

A particle is a very small part of a body, the volume of which is negligible, and hence can be represented by a point only. A rigid body has got a volume and is composed of a number of particles but the particles will always maintain the same relative positions under any circumstances whatsoever. This is nothing but a pure theoretical consideration. Actually we meet in nature with bodies, the composing particles of which change their relative positions due to the action of external agent. These bodies are called elastic bodies.

At first the portion of the science that deals with particles alone will be discussed. Though occasionally the word 'body' and things

representing bodies, such as, train, stone, etc., have been used, it is always to be remembered that they have been used in the same sense with a particle. Any deviation from this has been mentioned in proper time and place.

3. Fundamental quantities in Mechanics.

There are three fundamental quantities—length, time and mass. From the very childhood everyone of us is acquainted with the terms —length and time, and it is easy to form an idea of these two quantities. The term, mass, is quite unknown for the beginners and will be discussed in details in the proper place.

4. The Units.

Unit of length. The unit generally used in England is a Yard. The genuine standard measure of a yard as defined by the Act of Parliament, 1855, is the distance between the centres of the transverse lines on the two gold plugs in the bronze bar, now deposited in the Standards' Office of the Board of Trade at Westminster. The measure is taken at a temperature of 62° Fahrenheit.

Mile, Foot, Inch, etc., are also chosen as units when required. These are nothing but the multiples and sub-multiples of a yard.

Another unit, a Metre, is recommended by a committee of the British Association for the Scientific Purposes. A Kilometre, a Centimetre, etc., are also used as units. They are the multiples and sub-multiples of a Metre.

The genuine standard measure of a metre was defined by a Law of the French Republic in 1795, as the distance between the ends of a platinum rod made by Borda. The measure is taken at the temperature corresponding to the melting point of ice.

Though the fundamental units of length are a Yard and a Metre in the two systems respectively, the primary units for mathematical treatment in Mechanics are generally taken as a Foot in the British system and a Centimetre in the Metric system.

Unit of time. The earth is moving round the sun. The interval between two successive passages of the sun across the meridian of any definite place differs slightly from day to day. The mean value of these intervals is divided into 86400 ($=24$ hrs.$\times 60$ min.$\times 60$ sec.) equal parts and each of these divisions is named as a Mean Solar Second and is the fundamental unit of time.

Other general informations.

TABLE I

	$90+\theta$	$90-\theta$	$180+\theta$	$180-\theta$
sin	$\cos\theta$	$\cos\theta$	$-\sin\theta$	$\sin\theta$
cos	$-\sin\theta$	$\sin\theta$	$-\cos\theta$	$-\cos\theta$

TABLE II

Angles	0	15	20	30	45	60	75	90
sin	0	.259	.342	.500	.707	.866	.960	1
cos	1	.960	.940	.866	.707	.500	259	0
tan	0	.268	.364	.577	1	1.732	3.732	∞

TABLE III — Sq. root up to 10

Number	1	2	3	4	5	6	7	8	9	10
Sq rt.	1	1.414	1.732	2	2.236	2.449	2.646	2.828	3	3.162

$\pi = 3.141562 \qquad \pi^2 = 9.8696 \qquad \frac{1}{\pi^2} = 0.1013$

$\sqrt{\pi} = 1.7725 \qquad \log \pi = 0.4972$

during computation take up to two places of decimal

$1^\circ = \frac{\pi}{180} = 0.0176$ radian, *i.e.*, to convert degrees to radians multiply degrees by $\frac{\pi}{180}$ or 0.0176

1 radian $(1^c) = \frac{180}{\pi} = 57.29$ degrees, *i.e.*, to convert radians to degrees, multiply radians by $\frac{180}{\pi}$ or 57.29

In a circle of curvature

Radius of curvature,
$$R = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Measure of lengths — relation between French and British systems.

1 foot	=	30.48 centimetres	1 metre	=	3.28 feet
1 yard	=	0.914 metre		=	39.37 inches
1 mile	=	1.6093 kilometre	1 kilometre	=	0.6213 mile
1 mile	=	5280 feet	1 nautical mile (1 knot)	=	6080 feet (one-sixtieth of a degree measured at the equator)
15 miles per hour	=	22 feet per second		=	1.152 mile

NOTATIONS

A, B, C, D, etc.	points, spaces, objects, angles.
A, a	area.
D, d	diameter, distance.
C	constant.
E, e	efficiency.
F	force.
G	centre of gravity.
H, h	height, altitude.
I	moment of inertia.
K	product of inertia.
L, l	length.
M	mass, moment of resistance, mechanical advantage.
N	number, rotational speed, normal reaction.
O, o	centre, origin.
P	force in general.
Q	quantity of work, force.
R	normal pressure, resistance, external radius of a hollow object, reaction, resultant.
T	time, torque, moment of a couple, tension.
V	volume, velocity.
W	load, weight (total and general).
X, Y, Z	rectangular co-ordinate planes and axes.
a, b, c, d, etc.	points, lengths, etc.
a, b, c,	sides of a triangle.
a	amplitude.
e	base of the Napierian log, co-efficient of restitution, stiffness of material.

f	linear acceleration.
g	acceleration due to gravity.
k	constant, radius of gyration.
m	mass.
n	number, rotational speed.
r	radius in general, internal radius of a hollow body.
s	space, half the perimeter of a triangle, distance, displacement.
t	time, temperature, tension, thickness.
u	initial velocity.
v	final velocity, velocity in general, volume.
w	weight, density, intensity of load.
x, y, z	rectangular co-ordinates.

Greek Alphabets.

α	Alpha	
β	Beta	
γ	Gamma	
δ	Delta	
θ	Theta	angles in general.
λ	Lamda	
ϕ	Phi	
ϵ	Epsilon	
μ	Mu	coefficient of friction.
π	Pi	circular measure of 180° (in radians).
ω	Omega	angular velocity in radians.
Σ	Sigma	sign of summation.
ρ	Rho	constant.

Alpha also represents angular acceleration and *delta* represents density. It is to be marked that the same alphabet indicates more than one thing. Actual denotation must be ascertained from the place of occurrence. Also it is to be noticed that generally the initials have been used for notations though there is exception. Subscripts will be used to denote co-ordinate axes and to differentiate the different natures of the same thing, *e.g.*, total and accelerating forces will be represented by P_T and P_f respectively.

PART I

CHAPTER II

KINEMATICS

5. Motion. If a particle changes its position it is said to be in motion. There are primarily two kinds of *motion* :

(1) Rectilinear, *i.e.*, motion in a straight path.

(2) Curvilinear, *i.e.*, motion in a curved path.

Rectilinear Motion — Straight Line Motion

6. Displacement. Displacement is the change of position of a particle in a definite direction. It is measured by the straight distance between the two positions—initial and final. It has, therefore, both the magnitude and direction. It will generally be represented by the letter 's'.

If a particle moves from A to C along the path ABC (Fig. 1-1), the displacement is AC, the straight distance in the direction from

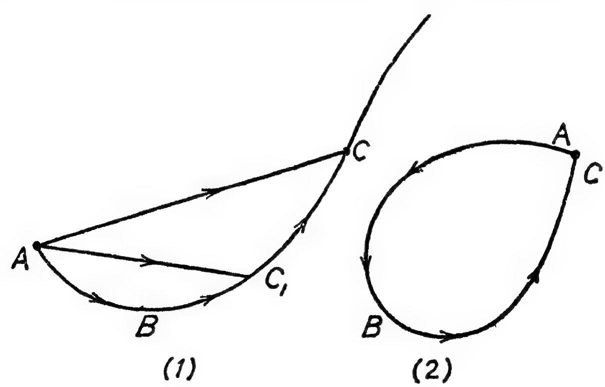


Fig. 1

A to C along the straight path AC. But ABC is the actual space traversed. Similarly when the particle moves along ABC to C_1, the displacement is represented by AC_1, the straight distance between A and C_1 in the direction from A to C_1. But if the particle moves

along a path ABC, shown in Fig. 1-2 —starting from the point A and reaching at the point C, which coincides with the point A—the particle gets no displacement at all.

7. **Speed.** The time rate of motion is the *speed*. It is independent of the direction. It may be uniform as well as varying. If a particle moves through equal lengths of its path in equal periods of time, the particle is said to move with uniform speed. In case of varying speed the total length of the actual path traversed divided by the total time required is the *time average speed*. In the previous article, if the time required to travel from A to C (Fig. 1-1) be 3 seconds and if the path ABC represents 15 feet, then the time average speed is $15 \div 3 = 5$ feet per second.

8. **Velocity.** The time rate of displacement of a moving particle, *i.e.*, displacement per unit time is the velocity of the particle. It is nothing but a multiple or sub-multiple of the total displacement and, therefore, like displacement has both magnitude and direction. The velocity may be uniform as well as varying. It will be represented by the letter 'v'.

9. **Uniform Velocity.** If the magnitude of the velocity remains constant throughout the time of travel, the velocity is said to be *constant or uniform*—otherwise it is called a *varying velocity*.

10. **Average Velocity.** In case of varying velocity, the average velocity of a moving particle is the total displacement of the particle divided by the total time taken. This is also called the *time average velocity*.

If the straight distance, *i.e.*, displacement, AC in the problem of the article 7 be 12 feet then the time average velocity is $12 \div 3 = 4$ feet per second. The time average velocity is always less than time average speed if the path of motion is not straight. Velocity is a term which should only be used in cases of straight motions. But there is deviation, though harmless, as will be noticed when used in place of the term *speed*.

If a particle moving with varying velocity undergoes a displacement of 100 feet in 5 seconds, then, $100 \div 5 = 20$ feet per second is the average velocity of the particle in the direction of the displacement. Again, if a particle moves with uniform velocity of 20 feet per second for 5 seconds, the displacement of the particle is $20 \times 5 = 100$ feet.

Thus, if in time t the total displacement of a particle, moving with uniform velocity v, be s, then, $s = vt$ or $v = \frac{s}{t}$. In this case the uniform velocity and the average velocity are the same.

11. Mean Velocity. *In case of varying velocity, half of the sum of the initial and final velocities during a period is the mean velocity when the variation is uniform, and in that case the mean velocity is equal to the average velocity.*

12. True Velocity. In case of varying velocity, the true velocity of a moving particle at any instant is the average velocity of the particle during a very small period of time including that instant.

13. True Velocity from the Displacement-time Curve. If the magnitudes of the time and displacement be represented along the two rectangular co-ordinate axes X and Y respectively and the points are plotted for displacement against time, the curve obtained by passing a smooth line through the points is the displacement-time curve.

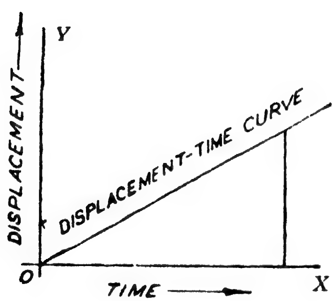


FIG. 2

When such curves are drawn it is found that in cases of uniform velocities the curves are similar to a straight lines as shown in Fig. 2 inclined to the horizontal axis passing through the origin. The velocity at any instant is represented in magnitude by the quotient of the ordinate and the abscissa corresponding to that instant, and this is nothing but the average velocity during the time of travel. In this case the ratio between the ordinate and the abscissa at every instant is constant, which means that the particle is moving with constant velocity. The average velocity here, is the true velocity at that instant.

If the velocity be not uniform the displacement-time curve will be other than a straight line. Let the curve be of the nature as is shown in Fig. 3. In such a case the true velocity at any instant is found out in the following way.

Say, the velocity at T or after the period OT is required to be

found out. Take any other point M very near to T. Draw two perpendiculars at T and M cutting the curve at Q and L respectively. Through Q draw a straight line QS parallel to TM cutting LM at S. Then in the period TM the displacement is $ML - TQ = SL$. Therefore, the average velocity during the period is

$$\frac{SL}{TM} = \frac{SL}{QS} = \tan \angle$$

i.e., equal to the tangent of the angle which the chord QL makes with the horizontal line QS.

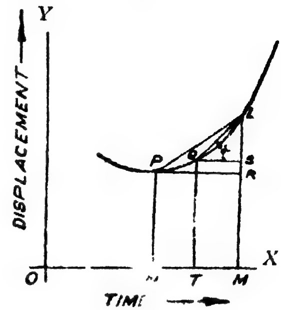


FIG. 3

If the interval of time is gradually reduced to an indefinitely small quantity, M will gradually approach towards T and the chord QL will gradually become the tangent line at Q. Hence, according to the definition, the true velocity at any instant can be represented in magnitude by the slope of the tangent line to the displacement-time curve, *i.e.*, the slope of the curve at that instant. Whether the velocity is increasing or decreasing that depends on the nature of the slope—upwards or downwards. Upward slope indicates the velocity in one sense while the other one in the opposite sense.

If the curvature of the line be not too sharp practical difficulties may arise to measure the inclination of the tangent line. In such a case, therefore, the best way to find out the slope of the tangent line is to take two equidistant points M and N on either side of the point T, as shown in Fig. 3. If NP be perpendicular drawn from N, then M and N being at equal distances from T, the chord PL will be very approximately parallel to the tangent line at Q. Now the measurement of the slope of the tangent line becomes easier. The angle LPR (PR being drawn parallel to the horizontal axis cutting ML at R) being equal to the angle made by the tangent line at Q with the horizontal axis,

$$\tan LPR = \frac{LR}{PR} = \frac{LR}{NM}.$$

Hence the magnitude of the true velocity at T is represented by $\frac{LR}{NM}$.

In the form of Calculus the slope can be represented by $\frac{ds}{dt}$,

where ds represents the small displacement in small time dt. Thus, the general form of the equation stands as, $v = \frac{ds}{dt}$.

14. Scale of the Diagram. To obtain the actual value of the magnitude of the velocity from the ratio obtained in previous cases the method is to multiply the ratio by the scale of the diagram. The scale of the diagram is the velocity represented by the slope of one unit of length in the horizontal direction and one unit of length in the vertical direction—both the units must be taken in the same scale. If this velocity be y units per second and the ratio between the ordinate and the abscissa be x, then the magnitude of the velocity is xy units per second. If the time-axis represents 'a' seconds for 1 inch and the space-axis represents 'b' feet for 1 inch, then the velocity represented by a slope of $1'' \times 1'' = \frac{b}{a}$, *i.e.*, y and if the slope of the tangent line be $\frac{c}{d}$, *i.e.*, x, (c and d are measured in the same unit), then the magnitude of the velocity $= xy = \frac{b}{a} \cdot \frac{c}{d} = \frac{bc}{ad}$ ft. per second.

15. To find out the Displacement from the Velocity-time Curves. These curves are drawn in the same way as the displacement-time curves were drawn. The only difference is that the vertical axis represents velocity instead of diplacement.

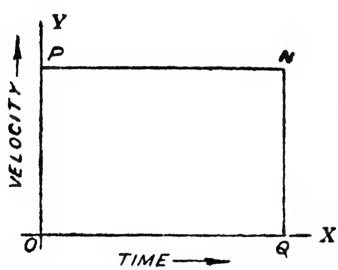


FIG. 4

Case I. Suppose the velocity is a constant one. The curve will then be a straight line parallel to the time (X) axis as shown in Fig. 4. It has already been obtained that if the magnitude of the uniform velocity be v and the period of travel be t, then the displacement is vt. In this diagram v is represented by OP along the Y (velocity) axis and the time t by OQ along the X (time) axis. Then, $vt = OP \times OQ =$ the area $OPNQ$. Thus, the area under the curve represents the total displacement in time t.

Case II. If the velocity be not uniform, and changes from v_1 to v_2

in time t, with a uniform change per unit time, the curve will be a straight line inclined with the time-axis, say, as shown in Fig. 5 (Explained in Chapter IV). Let v_1, v_2 and t be represented by OP, QN and OQ respectively. Then, v at any instant at a distance x from O along OX

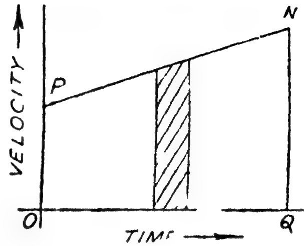


FIG. 5

$$= v_1 + \frac{x}{t}(v_2 - v_1)$$

—Geometry & Proportionality

$$\therefore \quad s = \int_0^t v.\,dt. = \int_{x=0}^{x=t} v.\,dx$$

$$\therefore \quad s = \int_0^t \left\{v_1 + \frac{x}{t}(v_2 - v_1)\right\}dx = \frac{1}{2}(v_1 + v_2)\,t \quad \cdots\cdots Eq.\ 1$$

i.e., $= \frac{1}{2}(OP + QN)\,OQ$, which is the area under the velocity-time curve.

Alternative Proof

Divide OQ into n number of small equal parts, each of which is, then, equal to $\frac{OQ}{n}$. If the change of velocity (which is an increment in this case as is evident from the diagram of Fig. 5) in each of these small periods be f, then, the total change is $(QN - OP) = nf$. The initial velocities during these consecutive small periods of time will, then, be represented by OP, $OP + f$, $OP + 2f$, $\cdots\cdots$ $OP + (n - 1)\,f$ respectively. Now, if n be too big a number, these initial velocities may be considered to be uniform throughout these successive small periods of time. If the total displacement be represented by S, then, $S = s_1 + s_2 + s_3 + \cdots + s_n$, where s_1, s_2, s_3 etc. are the displacements during the consecutive small periods of time respectively.

$$\text{Then, } S = OP.\,\frac{OQ}{n} + (OP + f)\frac{OQ}{n} + (OP + 2f)\frac{OQ}{n} + \cdots\cdots\cdots\cdots + \{OP + (n - 1.)f\}\frac{OQ}{n}$$

$$= n.\,\frac{OQ}{n}.\ OP + \frac{OQ}{n}.\ f\{1 + 2 + 3 + \cdots\cdots\cdots + (n - 1)\}$$

$$= OP.\,OQ + \frac{OQ}{n}.\,f.\,\frac{n(n-1)}{2} = OP.\,OQ + \frac{n-1}{2}f.\,OQ,$$

where n is very big, n is approximately equal to $(n - 1)$. Therefore, $(n - 1) f$ may be taken as nf, which is equal to $QN - OP$. Substituting this value of $(n - 1) f$ in the foregoing expression, S is represented by

$$OP.OQ + \frac{QN - OP}{2}.OQ = \frac{OP.OQ}{2} + \frac{QN.OQ}{2}$$
$$= \tfrac{1}{2}(OP + QN).OQ$$

= the mean altitude × the base of the diagram, which is nothing but the area under the curve PN.

Thus, in this case too it is found that the area under the curve represents the total displacement. Now, as the altitude represents the velocity, the mean altitude will represent the mean or average velocity. Hence, in case of uniformly varying velocity the mean velocity can also be defined as half the sum of the initial and final velocities. Thus, the mean velocity $= \frac{v_1 + v_2}{2}$, where v_1 and v_2 are the initial and final velocities respectively and it is equal to

$$\frac{S}{t}, \quad \text{or,} \quad S = \tfrac{1}{2}(v_1 + v_2)t$$

Case III. If the change in the previous case be not uniform, the curve will then be, say, as shown in Fig. 6. In this case too, $s = \int_0^t v.dt$, which is nothing but the area under the velocity-time curve.

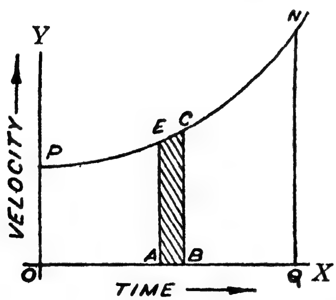


FIG. 6

Alternative Proof

Divide the time-axis into a large number of equal segments. Let AB be one of such segments and as A and B are very near to each other, the portion of the curve EC, between the verticals AE and BC, may be taken as a straight line. During the time represented by AB the velocity does neither remain constant as AE nor as BC, but varies from AE to BC uniformly—EC being considered as a straight line. Therefore, the displacement during AB, according to the case II, is represented by the area under the curve EC, which is equal to $\frac{1}{2}(AE + BC) AB$. In this way the area under the curve for each segment is found out. If they be equal to a_1, a_2, a_3, etc. respectively, the total area under

the whole curve, $A = a_1 + a_2 + a_3 \cdots\cdots$ and this represents the total displacement.

Thus, from the above various cases it is concluded that the area under the velocity-time curve represents the displacement.

It should be noted that in all these cases the areas under the curves must be multiplied by the scale of the diagram to obtain the actual value of the displacement. The scale of the diagram is the displacement represented by one square unit of the area under the curve. For example, in any of the previous diagrams if 1 in. along the Y axis represents a velocity y units per second and 1 in. along the X axis represents x seconds, then 1 sq. in. of the diagram will represent xy units of displacement. This is the scale of the diagram. Now, if the area $OPNQ$ (Fig. 6) be z sq. in., the total displacement is xyz units.

Illus. Ex. 1. *A motor car starting from rest describes s feet in t seconds from starting as given in the following table:—*

t	0	6	14	21	28	36	42	50	58	68
s	0	50	150	325	500	800	1100	1640	2190	2700

Find the average velocity of the car during 70 seconds starting from rest. Also find the true velocities at different moments and draw the velocity-time curve.

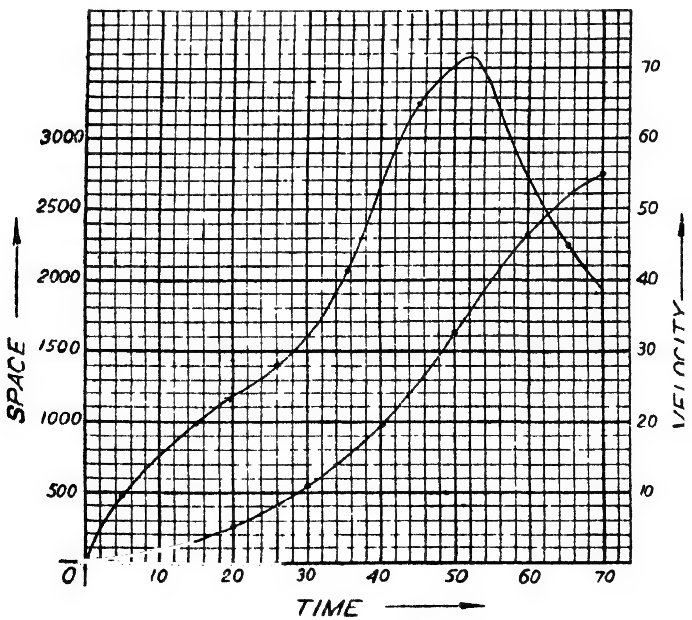


FIG. 7

On a piece of graph paper draw the displacement-time curve with the data given in the above table (Fig. 7). Scale chosen is one side of a big square representing horizontally 10 seconds and vertically 500 feet.

From the curve the following table is prepared to our advantage as will be clear from the procedure of the solution,

t	0	10	20	30	40	50	60	70
s	0	95	295	575	990	1640	2310	2760

Now, the average velocity during 70 seconds from starting is found out as follows:—

At the 70th second the reading of the space described is 2760 feet.
Therefore, the average velocity is $2760 \div 70 = 39.42$ feet per second.

Or,

At the 70th second the reading in terms of the side of a big square is vertically 5.52 and horizontally 7. Therefore the average velocity is represented by the ratio $\frac{5.52}{7}$. But the scale of the diagram $= 500 \div 10$, *i.e.*, 50 feet per second. Therefore, the average velocity $= \frac{5.52}{7} \times 50 = 39.42$ feet per second.

Now, the average velocity during the first 10 seconds, *i.e.*, the true velocity at the 5th second (Art 13) $= 95 \div 10 = 9.5$ feet per second.

Similarly, at the 15th second $= \frac{295-95}{10} = 20$ ft. per second

at the 25th second $= \frac{575-295}{10} = 28$ ft. per second

at the 35th second $= \frac{990-575}{10} = 41.5$ ft. per second

and so on.

Thus the results can be tabulated as follows:—

t	0	5	15	25	35	45	55	65
v	0	9.5	20	28	41.5	65	67	45

Now, the velocity-time curve is drawn with the above data. The velocity is represented along the Y axis and the time along the X axis. The scale chosen is, one side of a big square along the Y axis represents 10 feet and one side along X axis represents 10 seconds.

Alternative Method. (Differential Curve Method).

First choose the scale.

Take, 1″ vertical (left axis) = 1,000 feet —space scale.
1″ vertical (right axis) = 20 feet —velocity scale.
1″ horizontal (from origin) = 20 seconds —time scale.

Next, with the data given draw the space-time curve (Fig.8). Produce the horizontal axis to the left as shown and measure a length, *OP*, from the

origin, which is called the *Pole-distance*. The pole-distance is determined in the fololwing way,

$$\text{Pole-distance} = \frac{\text{Old vertical scale}}{\text{New vertical scale}}$$

$$= \frac{1000}{20} = 50 \text{ units horizontally,}$$

1" · horizontally = 20 units. Therefore, the pole-distance is taken as $50 \div 20 = 2.5$ inches.

Now, divide the time-axis into a large number of convenient small parts according to the nature of the curve—here, say, 8 seconds, *i.e.*, 0.4 inch is the measure of each part. It is not indispensable that each part should be equal. Join the ends of the portions of the curve for these parts. From the pole *P* draw straight lines parallel to these straight lines cutting the nearest vertical axis (the left) at different points. Draw horizontal lines from

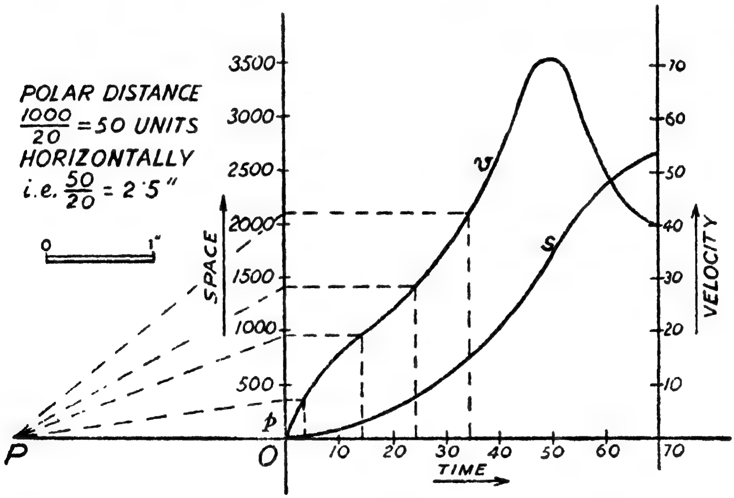


FIG. 8

these points, cutting the vertical lines through the mid-points of period-length, as shown. Pass a smooth curve through the cutting points, and this curve represents the *velocity-time curve*. The ordinates of the curve represent velocities at different instants in a definite scale. An ordinate multiplied by the scale of the diagram is the actual value of the magnitude of the velocity.

Here, the velocity-time curve is called the *differential curve* derived from the *original* or the *primitive* curve, *i.e.*, the *space-time curve*.

It is to be marked that only the magnitudes of a velocity (and not the direction) at different instants can be read off from the curves of this type obtained either by the previous method or the present one.

The scale of the diagram $= \frac{a}{bc}$, where a is the space scale, b is the time scale and c is the pole distance in inches.

Here the scale is $\frac{1000}{20 \times 2.5} = 20$ feet/sec.

That the scale is $\frac{a}{bc}$, can be proved as follows:

$\frac{ds}{dt} = v =$ slope of the curve $\times$ the scale of the slope.

$= \text{slope} \times \frac{a}{b}$, where, a units are represented by 1" vertically, and b units are represented by 1" horizontally

$\therefore$ slope $= v \, . \, \frac{b}{a}$

or $\frac{Op}{OP} = \tan pPO$, or, Op (ordinate) $= OP \tan pPO$

$= OP \times \text{slope}$

$= OP. \;\; v. \frac{b}{a} = \frac{bc}{a}. \, v$

where c is the pole-distance in inches.

From which, $v = Op \frac{a}{bc}$, $\frac{a}{bc}$ is the scale of the diagram.

Illus. Ex. 2. *A tram car starting from rest attains a speed of v feet per second in t seconds as given in the following table —*

t	0	4	10	14	18	24	30	34	42	50	60
v	0	12	21.5	26	30	34	37	37.4	37	35	31

Find the distances covered from the starting point at different moments during 60 seconds and draw the displacement-time curve.

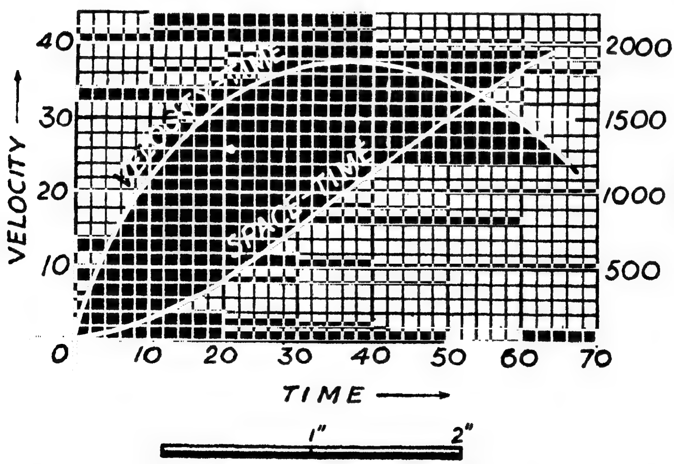


FIG. 9

Draw the velocity-time curve with the above data just in the same way as was done in Example 1. From the curve prepare the following table (at

regular intervals of 10 seconds starting from the 5th second) as shown in Fig 9

t	0	5	15	25	35	45	55
v	0	14	27 3	34 7	37 8	36 2	33

The velocities at the 5th, 15th, 25th, etc, second may be taken as mean velocities during the first, second, third, etc, consecutive periods of 10 seconds respectively Therefore, the displacement in the 1st 10 seconds $= 14 \times 10 = 140$ feet

Similarly, the distance covered in the 2nd 10 seconds is equal to the velocity at the 15th second multiplied by 10, *i e* $27\ 3 \times 10 = 273$ ft Therefore, the total distance after 20 secs $= (140 + 273) = 413$ ft Proceeding in the same way the distances covered in 30, 40, 50 and 60 seconds are found out and the results are tabulated as follows —

t	0	10	20	30	40	50	60
s	0	140	273 + 140 = 413	347 + 413 = 760	378 + 760 = 1138	362 + 1138 = 1500	330 + 1500 = 1830

Now, the curve is drawn as shown in the diagram (Fig 9) with the above data The scale is chosen just as in the case of Example 1

Note that the total displacement is found out in this case by a gradual method starting from the beginning

The total displacement, 1830 feet, can be obtained all at once from the curve Find the area under the curve by an instrument called planimeter or otherwise It is found to be 4 575 sq in approximately Now, one side of a big square of the graph paper measures ½ in Therefore, 1 sq in of the area under the curve represents $20 \times 20 = 400$ feet Hence the total displacement is equal to $4\ 575 \times 400 = 1830$ feet

Second Method (Integral Curve Method)

At first choose the scales for velocity, space and time

Let 1″ of the left vertical axis $= 20$ feet (velocity scale)

1″ of the right vertical axis $= 1{,}000$ feet (space scale)

and 1″ of the horizontal axis $= 20$ seconds (time scale)

Draw the velocity-time curve Next, just in the same way as was done in case of drawing the differential curve, take the pole, P In this case the

$$\text{pole-distance, } c = \frac{\text{New V-scale}}{\text{Old V-scale}} = \frac{1000}{20} = 50 \text{ units horizontally}$$

$$1'' \text{ horizontally} = 20 \text{ units}$$

$$\therefore \text{pole-distance must be made, } \frac{50}{20} = 2\ 5 \text{ inches}$$

. Now, divide the horizontal axis in small segments, say, 0.4 inch each. From the middle points of these segments draw vertical straight lines (dotted as shown, Fig.10) cutting the velocity-time curve at different points. From these points draw horizontal lines cutting the left hand vertical axis at points, *p*, *q*, *r*, etc. respectively, which are in turn joined with the pole *P*.

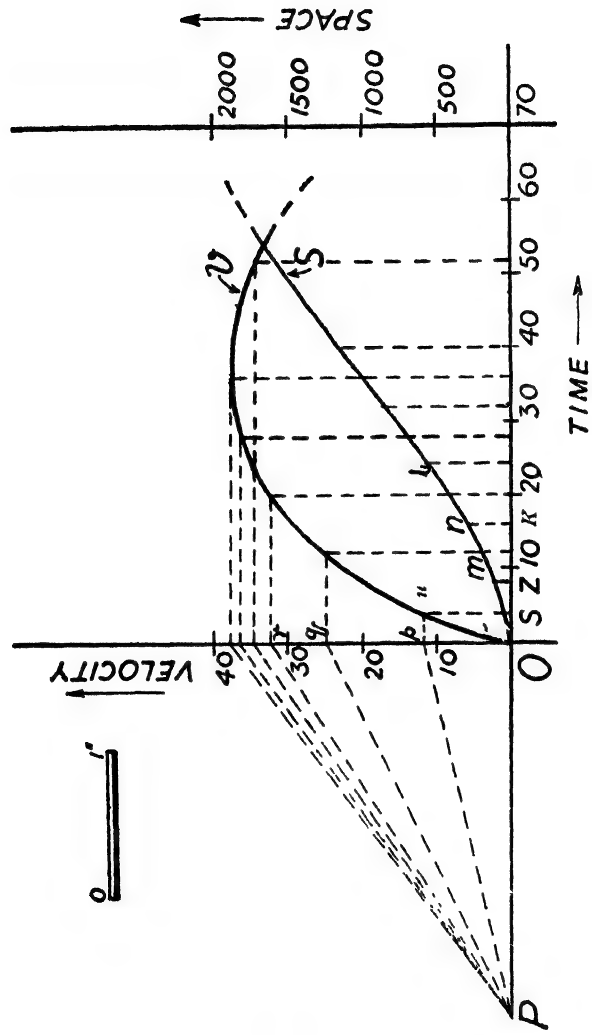


FIG. 10

From the origin draw a straight line Om parallel to Pp cutting the vertical through the end of the first segment at m; from m draw mn parallel to Pq cutting the vertical through the end of the second segment at n, and so on. Pass a smooth curve through the points m, n, l, etc. thus obtained. This curve is the space-time curve.

The velocity-time curve is called the *original* or *primitive* curve and the space-time curve is called the *sum-curve*. The space-time curve is the *integral curve* derived from the velocity-time curve.

Space described is represented by the area under the velocity-time curve. Total space described in time, $T (= n.t$, where n is too large a number), $= v_1t + v_2t + v_3t + \cdots$, where v_1, v_2 v_3, etc. represent the mean velocities during successive small intervals of t seconds respectively. Thus, the total space traversed $= \Sigma\ v.t$. Consider the first interval. From similarity of triangles, $\frac{zm}{Oz} = \frac{Op}{OP}$ or, $zm.\ OP = Oz.\ Op = Oz.\ su$, that is, the area under the curve, say, it is A_1. Then, $A_1 = zm.c$, where c is the pole-distance.

Similarly, it can be proved that for the first and second intervals, the area under the curve, *i.e.*, sum of the area for the 1st and that for the 2nd interval $= c.\,rn$, and so on.

Hence, the ordinates of the space-time curve represent the spaces during different periods in some definite scale.

Now, $A_1 = zm.c$. Again, the actual space traversed is equal to the area under the curve multiplied by the space scale, *i.e.*, 1″ vertical × 1″ horizontal representing $a.b$ feet.

Therefore, space $= A_1 \times ab = a.b.c.zm$. Hence, for the sum-curve the scale is abc. (1″ velocity axis represents a units and 1″ time-axis represents b units). Thus, space = ordinate in actual scale multiplied by the scale of the diagram. Scale of the diagram $= abc$. Here it is $20 \times 20 \times \frac{5}{2} = 1{,}000$ ft.

It is to be noted that in the differential curve as well as integral curve there is no definite rule for selecting the pole-distance. Any value may be taken, but for advantage the value should be chosen in such a way that the new scale becomes a round number.

PROBLEMS

1. From the motion of a train the following table is prepared. In t seconds the distance traversed by the train is s feet.

t	0	4	8	13	21	29	33	41	47	52	60
s	0	18	48	99	189	282	324	417	483	540	630

Find approximately the true velocities after 5, 15, 25, 35, 45 and 55 seconds from the beginning and plot the curve with these values against the time. Read from it the true velocity at the 60th second.

2. From the following velocity-time curve find the distance covered in 5, 15, 25, 35 and 45 seconds and draw the space-time curve

t	0	4	10	14	23	30	35	40	48	50
v	0	4	12 5	16	24	30 5	35 8	43 5	62	68 5

3. Draw the space time curve from the following equation and develop from it the velocity-time curve and read off the velocity after 8 seconds

$$s = 50 - 5\ t^2$$

CHAPTER III

VECTORS

16. Vector and Scalar Quantities. There are certain physical quantities which can be denoted by numbers only representing their magnitudes in terms of proper units chosen, such as, the speed of a particle, the length of a rod, the volume and temperature of a body, a sum of money etc. These quantities are called *scalar quantities*.

Other quantities, like displacement and velocity, which involve the idea of direction in addition to magnitude, cannot be denoted completely by numbers alone. Some additional description is required to express the direction. All such quantities which have both magnitude and direction are called *vector quantities*.

17. Vectors. A vector quantity can be represented by a straight line—the length represents the magnitude and the direction of the line denotes the direction of the quantity. Such a straight line is called a vector. A vector has a definite length and direction but no definite position in space. The direction can be indicated by lettering. For example, if a vector is named as AB (Fig. 11), the direction A to B indicates the direction of the quantity. It can also be denoted by an arrow-head in the line. In that case a single letter is sufficient to name the vector. The vector C (Fig. 11) represents fully the vector AB, because it is equal in length to AB and the arrow-head indicates the direction A to B. There is also a custom to adjust the arrow-head at the finishing end of the straight line C_1 (Fig. 11). It is to be noted that the vector BA is not same as the vector AB. Though they are equal in magnitude but are opposite in direction.

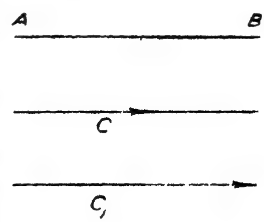


FIG. 11

Actually a vector quantity has three elements—magnitude, direction and sense. If a body is thrown with a velocity of 50 feet per second making an angle of 60° with the surface of the earth upwards, 50 feet per second is the magnitude, at an angle of 60° with the surface of the earth is the direction, and upwards is the sense.

Ordinarily by the term 'direction', both the sense and direction will be included.

18. Addition of Vectors. To add vector CD with vector AB (Fig. 12). Take any point a. From a draw a straight line ab to represent vector AB in magnitude (in some definite scale) and direction. From b (not from a) draw a straight line bc to represent CD in magnitude (in the same scale) and direction. Join ac. Then ac (not ca) represents the result of the vector addition in magnitude

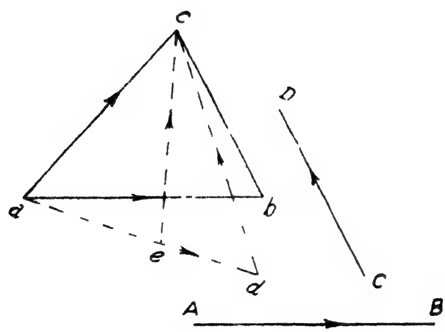


FIG. 12

and direction. Symbolically, $AB + CD$, *i.e.*, $ab + bc = ac$. The vector ac is said to be the resultant vector and ab and bc are said to be the components of ac. This same vector ac may be called to be the resultant of any two other components ad and dc, or ae and ec (Fig. 12). That is to say, any vector can be resolved into innumerable pairs of component vectors.

19. Subtraction of Vectors. Suppose the vector CD (Fig. 13) is to be subtracted from the vector AB. Take any point a. From a draw (similar to the previous case) a straight line ab to represent the vector AB in magnitude and direction. From b draw a straight line bc to represent CD in magnitude but opposite in direction (Fig. 13-1). Join ac. Then ac is the result of the vector subtraction. Mark the diagram. The result of the vector addition is ac', but the result of the vector subtraction is ac. In this particular diagram ac is greater than ac' in length, *i.e.*, the result of the subtraction has bigger magnitude, but this is not true for each and every case. Whether the magnitude of the result of the vector

subtraction or that of the result of the vector addition will be greater, that depends on the directions of the vectors.

Note that the direction of the result of a vector operation is always towards the last vector drawn in succession in the diagram of each case.

Alternative Method.

Take a point *a* (Fig. 13-2), and from *a* draw two vectors *ab* and *ac* to represent *AB* and *CD* in magnitude and direction respectively.

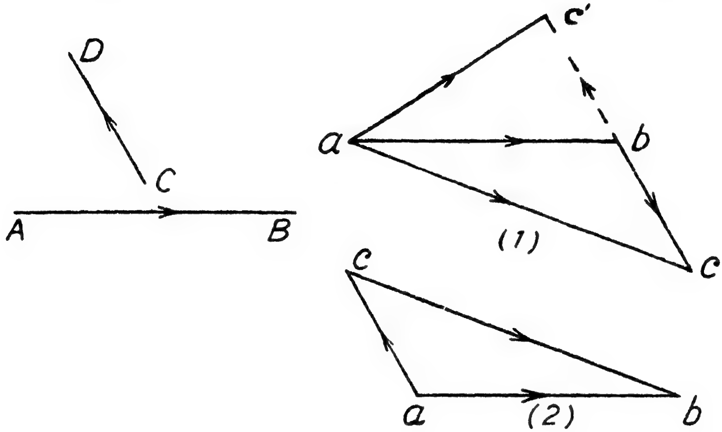


FIG. 13

Join *cb*. Then, *cb* represents the vector difference in magnitude and direction. It is to be marked that the direction is towards the vector from which the difference is determined.

The two methods should be compared.

20. Resultant and Component Velocities. It is very often found that a particle possesses two or more velocities simultaneously. For example, when a man walks from one end to the other of a running tram car, the man has two velocities simultaneously—one, due to the velocity of the tram car and the other, his own. In such cases where there are simultaneously more than one velocity in a particle, the particle is found to move with a single velocity which is the effect of all the individual velocities. These individual velocities are called the *component velocities* and the velocity with which the particle is found to move is called the *resultant velocity*.

21. Resolution and Composition of Velocities.

(*a*) *Vector Addition.* Velocity is a vector quantity. All the individual velocities are added vectorially and the resultant is found out. Suppose a particle have five simultaneous velocities, V_1, V_2, V_3, etc., in the same plane as represented in the diagram (Fig. 14-*a*). It is to be noted here that it is not absolutely necessary that the velocities must be taken in the same plane but for the present we shall consider the cases of velocities in one plane only. Take any point *a* (Fig. 14-*b*). From *a* draw a straight line *ab* to represent V_1 in magnitude and direc-

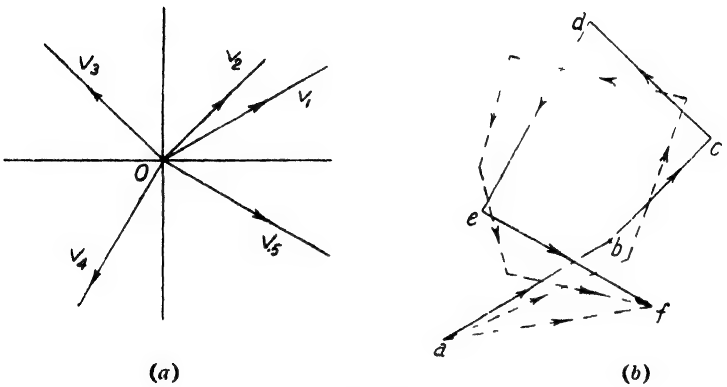


FIG. 14

tion. From *b* draw *bc*, a straight line in order to represent V_2 in magnitude and direction in the same scale. Similarly, draw *cd* from *c*, *de* from *d* and *ef* from *e* respectively in order to represent V_3, V_4 and V_5. Join *af*. Then *af* is the result of the vector addition, *i.e.*, represents the resultant velocity of the particle. The other sides of the diagram represent the components of the resultant *af*. In drawing the diagram it is immaterial whether one vector cuts another (Fig. 14-*b*, *ef* cuts *ab*). Similar diagrams which contain more than three sides are called the polygon of velocities. Any side of the polygon *abcdef* may be said to be the resultant of other components if the signs are properly adjusted. Also the same resultant *af* may be made to be the resultant of five or more velocities other than represented by *ab*, *bc*, *cd*, *de* and *ef* (as shown in dotted diagram, Fig. 14-*b*). Thus any vector can be resolved into innumerable components.

(*b*) *Triangle of Velocities.* If two simultaneous velocities imposed in a particle can be represented fully by two sides of a triangle

in order, the third side will represent their resultant if the sign be properly adjusted as described above. Such a triangle is called a triangle of velocities. The diagram is nothing but a diagram for vector addition of two vectors. The magnitude of the resultant can be calculated from the trigonometrical relation

$R = \sqrt{V_1^2 + V_2^2 + 2V_1V_2 \cos\theta}$, where R is the resultant and V_1 and V_2 are the component velocities respectively and θ is the angle between the two component velocities.

(*c*) *Parallelogram of Velocities*. This is another method of finding out the resultant of two simultaneous velocities.

If two velocities be fully represented by two adjacent sides of a parallelogram then the diagonal of the parallelogram passing through the point of intersection of two adjacent sides, having sign properly adjusted for the direction, represents the resultant velocity. From

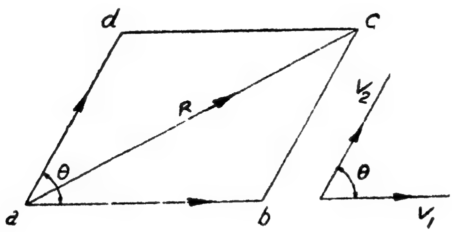


FIG. 15

Fig. 15 it is found that the diagonal ac is nothing but the result of the vector addition $ab + bc$, *i.e.*, $V_1 + V_2$. In this case too, the magnitude of the resultant can be obtained from the relation (Trigonometry)

$$R = \sqrt{V_1^2 + V_2^2 + 2V_1V_2 \cos\theta},$$

where notations are the same as in the previous case. In cases of (*b*) and (*c*) the direction, where the actual vector diagram is not drawn, is found out from (solution of triangles)—

Sides of a triangle are proportional to the sines of the opposite angles.

The statement made above (*c*) can be directly proved in the following way: Suppose a moving particle has two velocities simultaneously—10 miles per hour eastward and 8 miles per hour in the direction 60° north of east (Fig. 16). Let the body start from the point O. After 15 minutes the body is shifted by $\frac{10}{4}$ *i.e.*, 2.5 miles (OA_1) in the eastward direction and at the same time it has got a displacement in the direction of the other velocity by an

amount equal to $\frac{8}{4}$, *i.e.*, 2 miles (A_1P_1), occupying the position P_1. Similarly, the body will occupy the positions P_2 ($OA_2 = 2.5 \times 2$ miles, $A_2P_2 = 2 \times 2$ miles), P_3 ($OA_3 = 2.5 \times 3$ miles, $A_3P_3 = 2 \times 3$ miles), P_4 ($OA_4 = 2.5 \times 4$ miles, $A_4P_4 = 2 \times 4$ miles) after 30, 45 and 60 minutes respectively. It is evident from the positions of the points that if O, P_1, P_2, P_3

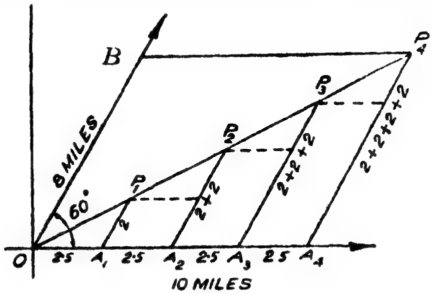


FIG. 16

and P_4 be joined together they lie in a straight line, which is nothing but the diagonal of the parallelogram OA_4P_4B The adjacent sides of the parallelogram, OB and OA_4 are representing the two component velocities respectively and OP_4, the diagonal passing through the point of intersection of OB and OA_4 represents the resultant.

(d) Another method of obtaining the resultant of two or more simultaneous velocities—*the method of resolution.*

It has already been proved that a vector can be resolved into any number of components. The method adopted here is to resolve each of the component velocities into two other components in two definite directions at right angles to each other (preferably in the horizontal and vertical directions), say along two rectangular co-ordinate axes OX and OY.

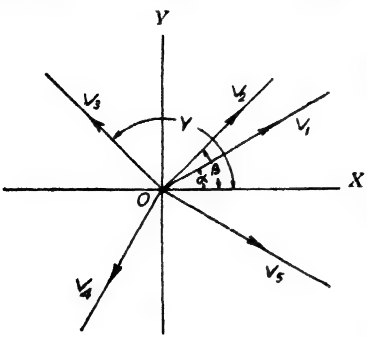


FIG. 17

Suppose V_1, V_2, V_3, V_4, etc., are the component velocities (Fig. 17). Each of them is resolved into two components in the directions along the horizontal and vertical axes OX and OY respectively.

$$\text{Then}\quad \left.\begin{aligned} V_1 &= V_1 \cos\alpha + V_1 \sin\alpha \\ V_2 &= V_2 \cos\beta + V_2 \sin\beta \\ V_3 &= V_3 \cos\gamma + V_3 \sin\gamma \end{aligned}\right\} \text{Geometric Sum}$$

and so on.

Now, add all the horizontal and vertical components separately and let them be represented by ΣH and ΣV respectively. Then,

$$\Sigma H = V_1 \cos \alpha + V_2 \cos \beta + V_3 \cos \gamma + \cdots\cdots$$
$$\Sigma V = V_1 \sin \alpha + V_2 \sin \beta + V_3 \sin \gamma + \cdots\cdots$$

Now, if R be the resultant, then, $R = \sqrt{\Sigma H^2 + \Sigma V^2}$; and the direction is such that $\tan \theta = \frac{\Sigma V}{\Sigma H}$, where θ is the angle made by the resultant with the horizontal direction. ΣH and ΣV are also denoted by ΣX and ΣY respectively, which indicate the sum of the components along X and Y axes respectively.

N.B. It is to be noted here that all the motions considered are co-planer. In this elementary treatise of science generally the co-planer motions will be treated, except otherwise mentioned. It is also to be marked that if simultaneous velocities are imposed on a particle and the particle does not move, the result of the vector addition must be zero, *i.e.*, the vectors, representing the velocities, drawn in order will form a closed figure. Also, if those velocities be resolved into two directions along the two rectangular co-ordinate axes, OX and OY, both H and V must be equal to zero, where ΣH and ΣV represent the sum of the components along the X and Y axes respectively.

Illus. Ex. 3. *The maximum speed of a swimmer in still water is 2.5 miles per hour. Find the velocity of the man in magnitude and direction when he crosses a stream heading 60° with the current which flows at a velocity of 5 miles per hour.*

Method I. Vector addition. Scale 1 in. = 2.5 miles. Take any point a (Fig. 18). From a draw a straight line ab 2 in. long horizontally to represent the velocity of the current. From b draw bc a straight line 1 in. in length making an angle 60° with the horizontal direction. Join ac. Then ac represents the resultant velocity in magnitude and direction. Measure ac.

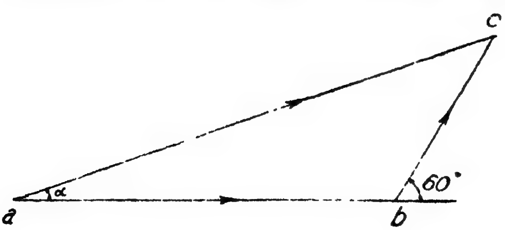


FIG. 18

Then, ac is found to be 2.65 in. long. Therefore ac represents $2.65 \times 2.5 = 6.62$ miles approximately. By actual measurement the angle cab is found to be 19° approximately.

Method II. Triangle of velocities In the above diagram it is found that *abc* is a triangle in which *ab* and *bc* are representing the components and *ac* is their resultant In the triangle then,

$$(ac)^2 = (ab)^2 + (bc)^2 - 2ab\,bc \cos abc = (ab)^2 + (bc)^2 + 2ab\,bc \cos 60$$

or $$ac = \sqrt{5^2 + (2\ 5)^2 + 2 \times 5 \times 2\ 5 \times\ 5} = 6\ 615$$

Method III Parallelogram of Velocities (Fig 19) *ab* represents the velocity of the current and *ad* the swimmer s speed The resultant velocity is then represented by the diagonal of the parallelogram *abcd* passing through the point of intersection of *ab* and *ad*

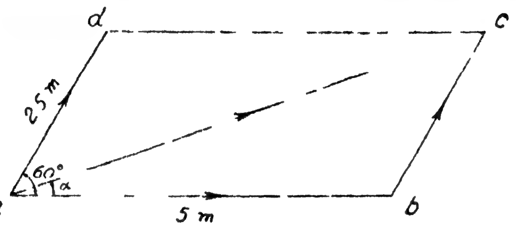


FIG 19

The diagonal $ac = \sqrt{(ab)^2 + (ad)^2 + 2\,ab\,ad \cos 60} = 6\ 615$

The direction in the previous case as well as in this case can be found out, as follows, in the triangle *abc*

$$\frac{ac}{\sin abc} = \frac{bc}{\sin cab} \quad \text{or} \quad \frac{6\ 615}{\sin 120} = \frac{2\ 5}{\sin \alpha},$$

where α is the angle made by the resultant velocity with the direction of the current

Thus, $\sin \alpha = 327$ From the table α is found to be 19° approximately

Method IV Method of resolution The velocity of the current has no vertical component The velocity of the man is then resolved to $2\ 5 \sin 60$ and $2\ 5 \cos 60$—the vertical and the horizontal components respectively Then the sum of the vertical components

$= 2\ 5 \sin 60 + 0 = 2\ 165$ and the sum of the horizontal components

$= 2\ 5 \cos 60 + 5 = 6\ 25$

Therefore the resultant $R = \sqrt{6\ 25^2 + 2\ 165^2} = 6\ 615$ miles/hr

Again, $\tan \alpha = \dfrac{2\ 165}{6\ 25} = 346$

From the table, $\alpha = 19°$ approximately

Illus. Ex. 4. *A man is travelling by a train which runs at a speed of 10 metres per second eastward He finds a jackal standing near the railway line and throws a stone to strike it when the beast is at an angle of 30° due north with the line from him If the velocity of the stone is 25 metres per second, find the direction of the throwing*

Suppose the train moves along *AB*. At the moment of throwing let the man occupy the position *A* and the beast *C* (Fig. 20). The stone has two velocities simultaneously—one due to the motion of the train and the other, its own motion. These two speeds combining together will move the stone

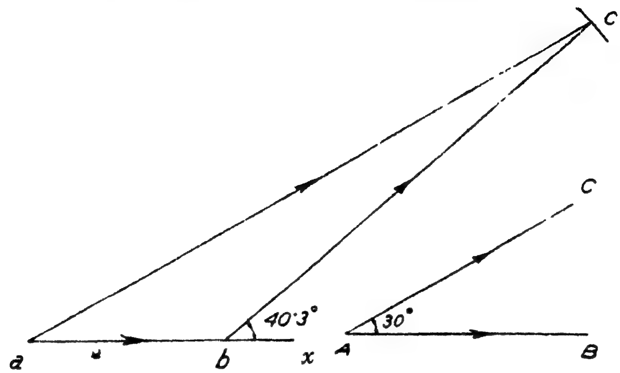


FIG. 20

along the path of *AC*. Thus if the vector diagram is drawn with the two component velocities and their resultant, the diagram will be a triangle. In the triangle, of the three sides,

1. One is given in length and direction (velocity of the train)

2. The length of the second one is given but the direction is unknown (magnitude of the velocity of the stone)

3. Only the direction of the third one, *i e.*, of the resultant is given.

Construction: Scale 1 in. = 10 metres.

Take any point *a* (Fig. 20). From *a* draw a straight line *ab*, 1 in. long to represent the velocity of the train. From *a* (not from *b*) where the resultant vector will end, draw a straight line *ac*, making an angle of 30° with *ab*. With *b* as centre and radius equal to a length, 2.5 in. representing the speed of the stone, draw an arc cutting the line *ac* at *c*. Join *bc*. Then *bc* gives the direction of throwing the stone. Produce the side *ab*, say, to *x*. Measure the angle *cbx* directly. The angle is found to be 40.3°. The direction of throwing is, then, 40.3° with the railway line due north.

22. Relative Displacement. When a particle changes its position from *A* to *B* (Fig. 21), the displacement *AB* is measured by the straight distance of the position *B* from the position *A*. Again, if the same particle is found to be shifted to another position *C*, the

total displacement is AC. Thus in every case the displacement is measured from the point A. Such a point is always needed for the measurement of displacement and is called the *point of reference*. This point of reference may be at absolute rest or in motion.

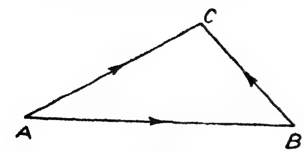


FIG. 21

In case where the point is at absolute rest the displacement is the absolute displacement of the particle. But if the point be in motion the displacement ascertains the position of the particle in relation to the point of reference. This displacement of the particle is called the *relative displacement* with respect to the point of reference.

23. Relative Velocity. The time rate of relative displacement of a moving particle with respect to another moving particle is the *relative velocity* of the former one with respect to the second.

As each and every particle in this universe is in motion the displacements and velocities of bodies, we generally deal with, are nothing but the relative displacements and relative velocities respectively.

Let two particles A and B (Fig. 22) have velocities V_1 and V_2 respectively. To find out the relative velocity of A with respect to B and *vice versa*.

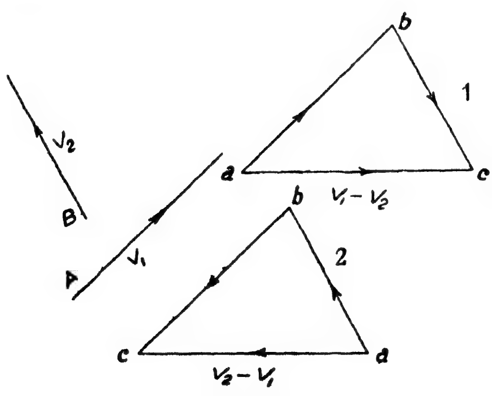


FIG. 22

Imagine the two particle to be isolated from the surrounding objects. If the relative condition of A is considered with respect to B,

A has got its own velocity V_1 and in addition to that the velocity of B has got an effect on A to move it with respect to B at the rate of V_2 just in the opposite direction to which B is moving. Thus A has got two simultaneous velocities, V_1 and $-V_2$ with respect to B. The relative velocity of A, then, is $V_1 + (-V_2) = V_1 - V_2 = ab + bc = ac$ (Fig. 22-1), *i.e.*, the velocity of B is subtracted from the velocity of A. It is to be noted here that when the effect of the velocity of B on A is added to the velocity of A, the particle B must be considered to be at rest in relation to A.

Similarly, the relative velocity of B with respect to A is the velocity of B minus the velocity of A, *i.e.*,

$$V_2 - V_1 = ab + bc = ac \qquad \text{(Fig. 22-2)}$$

Thus the relative velocity of a moving particle with respect to another moving particle is the velocity of the first particle minus the velocity of the second one.

A popular instance of relative motion is found when one travels by a train. He finds that the distant villages are running just in the opposite direction. The reason is that the traveller has got the same velocity with the train, but when he is inattentive to his own motion, *i.e.*, when he forgets the existence of the motion of the train it will appear to him as if the distant objects are moving away in a direction opposite to that in which the train is running. Actually the distant objects are not moving. The motion observed is nothing but the relative motion of the objects with respect to the train. To forget the velocity of the train is to consider the train to be at rest, *i.e.*, to impose the effect of the velocity of the train to the state of motion of the distant objects.

Note: *The utility of vector addition is to get the resultant velocity, whereas the relative velocity is obtained by vector subtraction.*

Illus. Ex. 5. *Two steamers leave a station at the same time ; one steams north-east at $12\frac{1}{2}$ miles per hour and the other 30 degrees south of east at 15 miles per hour. Find the velocity of the second steamer relative to the first one. Find also the time in which the two steamers will be 80 miles apart.* (Fig. 23).

Relative velocity of the second with respect to the first is equal to the velocity of the second minus the velocity of the first.

Construction. Take two intersecting straight lines—one horizontal and the other vertical. The scale chosen is 1 in. = 10 miles (Fig. 24).

Now from the point of intersection O, draw a straight line Oa in the fourth quadrant making an angle of 30 degrees with the horizontal direction to represent the velocity of the second steamer in magnitude and direction.

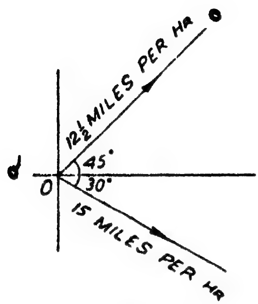


FIG. 23
Graphical representation of the data.

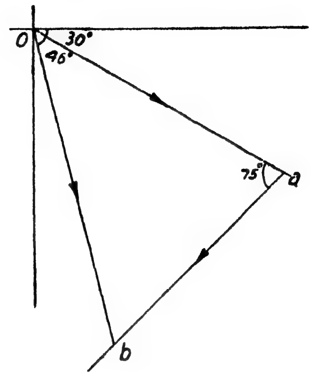


FIG. 24

Measure Oa as $1\frac{1}{2}$ in. From a draw a straight line ab $1\frac{1}{4}$ in. long in the direction opposite to that in which the first steamer moves (the angle $Oab = 45 + 30 = 75$ degrees) to represent the velocity of the first steamer but in the opposite direction. Join Ob; then Ob represents the velocity of the second steamer relative to the first one in magnitude and direction.

Measure Ob and the angle aOb. They are equal to 1.685 in. and 46 degrees respectively. Therefore, the relative velocity is equal to 1.685×10, *i.e.*, 16.85 miles per hour in a direction $(46 + 30)$, *i.e.*, 76 degrees south of east.

When we say that the 2nd steamer is running with this magnitude along Ob relative to the first one, we must take the first ship to be stationary at the point O. Therefore, the two steamers will be 80 miles apart in 80/16.85, *i.e.*, 4.75 hours.

Illus. Ex. 6. *At midnight a ship A is 25 miles due south of another ship B. A is running with a velocity of 20 miles per hour on a north-east course while B is proceeding towards east at 10 miles per hour. The two ships can exchange signals within an area of 10 miles radius. Find, when they can begin signalling and how long they can continue it. Also find, when the two ships will have the minimum distance between them.*

If the relative velocity of any one of the ships with respect to the other is found out, say, the relative velocity of B with respect to A is found out and is attributed to B, then A must be considered to be at rest at the position shown in the diagram (Fig. 25-A).

Construction. Scale 1 in. = 10 miles. Take two intersecting straight lines as in the previous case (Fig. 25-B). From the point of intersection O, measure Oa equal to $2\frac{1}{2}$ in. to represent 25 miles. From O take a length Ob 1 in. long

direction of the relative velocity as is shown in the diagram. With centre a and radius equal to a length 1.2 inch which represents the magnitude of the velocity of the second steamer, draw an arc cutting the straight line through b showing the direction of the relative velocity at c_1 and c_2. Join c_1 and c_2 with a. Then, ac_1 and ac_2 represent the two directions in which the second steamer may be moving in order to give the relative velocity a direction north-west. In the diagram, $ac_1 - ab = bc_1$, and $ac_2 - ab = bc_2$.

The angle c_1ab and the angle c_2ab measure 15° and 75.5° respectively. Therefore, the directions are 15° and 75.5° north of east. Again, $bc_1 = 0.468$ inch and $bc_2 = 1.672$ inch, therefore, the relative velocities are,

$$0.468 \times 10 = 4.68 \text{ miles}$$

and $1.672 \times 10 = 16.72$ miles respectively in the direction given.

PROBLEMS

4. Two simultaneous velocities—one eastward 10 miles per hour and the other 30° north of east having a magnitude of 12 miles per hour. Find the resultant. *Ans.* 21.25 miles per hour.

5. A steamer is crossing a river flowing east-west at a velocity of 10 miles per hour. If the vessel directs north-west and the propeller creates a velocity of 15 miles per hour, what is the resultant speed and course of the steamer? *Ans.* 23.17 miles per hour.
16.5° N of W

6. Find the resultant of the following three simultaneous velocities in a particle:—

   I. 5 miles per hour 30° N of E

   II. 15 miles per hour 30° W of N

   III. 12 miles per hour 30° S of W

   Ans. 16.5 miles per hour.
   35.5° N of W

7. A boy throws a stone with a velocity of 22 feet per second at right angles to the motion of a train running at the rate of 45 miles per hour. Find the magnitude and direction of the velocity of the stone relative to the train.

   Ans. 69.56 ft./sec. 18.45° with the motion of the train.

8. Three simultaneous velocities 5, 4 and 3 miles per hour are imposed on a particle, but it does not move. If the first velocity is eastward, find the directions of the other two. *Ans.* 36.9° N of W
53.1° S of W

9. A is moving eastward at 5 miles per hour while B moves along a path 30° west of north at 7 miles per hour. Find the relative velocity of A with respect to B and vice versa.

   Ans. 10.44 m.p.h. A — 35.5° S of E. B — just opposite.

10. A fighter plane is heading in a direction 60° north of east at a speed of 100 miles per hour in chase of an enemy fighter in the same altitude The wind blows in a direction 60° north of west at a speed of 50 miles per hour The enemy plane is fired at when it is at an angle of 45° north of east If the speed of the shot is 300 miles per hour, find the direction of firing *Ans* 40° N of E

11. A man tries to swim across a straight river along the least course If the velocity of the man in still water be 1 125 times the velocity of the flow of the river, find in what direction he should try to swim

Ans 143 2° against current

12 Rain appears to strike the ground at an angle of 45° from a train running at a speed of 30 miles per hour If the velocity of the train be 25 miles per hour when it touches the ground, find the direction of the rainfall

Ans 56° along the train motion

13 Two cyclists are running with speeds of u and $2u$ respectively in two straight directions making an angle θ between them Find the relative velocity of the second with respect to the other one

Ans $\sqrt{u^2(5-4\cos\theta)}$.

14 Two trains, each 500 feet long, are moving in parallel lines with velocities of 20 and 30 miles per hour respectively in opposite directions How long will they be in passing? *Ans* 13 63 seconds

15 Two ships are sailing from the same port in directions making an angle of 60° with each other If the relative velocity of one with respect to the other be 28 miles per hour and if one of the ships moves with a velocity of 20 miles per hour, find the magnitude of the velocity of the other

Ans 32 m p h

16. Two straight motor tracks cross each other On one of them a car is 10 miles from the crossing and is approaching towards it with a velocity of 50 miles per hour eastward On the other another car which is 15 miles away from the crossing, is running at a velocity of 60 miles per hour towards the crossing along a course 20° west of south Find when the two cars will have the minimum distance between them and also find the individual positions of the cars at that instant

Ans 14 04 secs

17. A cruiser runs at 25 knots due south At a certain instant a shot is fired with a muzzle velocity of 1500 feet per second at a stationary target lying on a line 20° north of east Find the bearing of the gun so that the shot hits the target *Ans* 21 5° N of E

Direction $v_o = 42\ 5$ ft /sec
Graphical solution should be avoided
Help—sol of triangle

18. A service steamer plies between two towns on the bank of the Hooghly river, 50 miles apart The trip up-stream takes double the time

required for down-stream trip. If a journey up and down with a stoppage of one hour takes 16 hours, find the velocity of the steamer itself and that of the stream. *Ans.* 7.5 m.p.h., 2.5 m.p.h.

19. Rain drops falling vertically strike the glass panel in front of the driver in an automobile, at an angle of 40° to the vertical. The automobile is running at a speed of 40 miles per hour. Find the absolute speed of the rain drops at the level of the point of strike. *Ans.* 69.92 ft./sec.

20. Two columns of infantry, covering a length of 4 furlongs, march with a constant speed. A messenger takes an instruction from the captain at the rear and running on a cycle at a speed of three times the speed of the marching columns delivers it to the captain leading the march and comes back to the former in 4.75 minutes. Find the speed of the cyclist assuming to be uniform throughout and the distance by which the columns advance by that time. Find also the space traversed by the messenger.

Ans. 15 m.p.h., 3 furlongs. 9 furlongs.

21. A submarine delivers a torpedo eastward to hit a cruiser which is moving along a course 60° north of west. If the speed of the torpedo and that of the cruiser be 15 and 25 knots respectively, find with what velocity the torpedo hits the cruiser. *Ans.* 35 knots.

22. Two railway lines cross each other at right angles. Two trains of equal length run at the rate of 30 miles and 45 miles respectively and approach the level crossing. If the distances of the two engines are 1500 feet and 2000 feet respectively from the crossing at the instant, find the greatest length that the trains can have to avoid collision between them.

Ans. 166.66 feet.

CHAPTER IV

ACCELERATION AND MOTION

24. If the velocity of a particle changes, the time-rate of the change of velocity is called the *acceleration.* Hence, it has like velocity, both magnitude and direction. It is, therefore, a vector quantity and can be represented by a vector. When the velocity increases the acceleration is considered as positive and when it decreases it is taken as negative and is termed as retardation. Acceleration and retardation may be uniform as well as varying. It is generally denoted by the letter 'f'.

25. **Uniform Acceleration.** If the velocity of a moving particle changes by an equal amount in each unit of time, the acceleration or retardation is said to be *uniform* or *constant* ; otherwise it is said to be *varying.* For example, if a particle, moving with a constant acceleration, undergoes a change in its velocity from, say, 5 feet per second to 30 feet per second in a period of 5 seconds, the total change of velocity in 5 seconds is (30 — 5), *i.e.,* 25 feet per second. Therefore, for each second the change is (25 ÷ 5), *i.e.,* 5 feet per second. Thus, the uniform rate of change of velocity or the uniform acceleration in this case is 5 feet per second *per second* ; and because the change is for the increment the direction of the acceleration is the same with the velocity. In the case where the velocity decreases, the direction of the retardation is just opposite to that of the velocity.

In the previous example, the initial velocity is 5 feet per second. After the lapse of 1 second it becomes, $(5 + 5)$ or 10 feet per second ; after 2 seconds it is $(5 + 5 + 5)$, *i.e.,* $(5 + \overline{2 \times 5})$ or 15 feet per second, after 3 seconds the value is $(5 + \overline{3 \times 5})$ or 20 feet per second and so on. It is evident from the foregoing results that the relation between the initial and final velocities follows the general equation. $y = mx + c$, the equation of a straight line. If the velocity-time curve be drawn, X axis representing the time and Y axis the velocity, it will be a straight line inclined to the X axis cutting the Y axis at a distance c from the origin, where c represents the initial velocity.

26. Average or Mean Acceleration. In the previous example the acceleration is constant as 5 feet per second per second throughout the whole period. But if the case be such that the velocity increases from 5 feet per second to 30 feet per second with varying acceleration according to the following table,

Time period in seconds	1st	2nd	3rd	4th	5th
Change per second	2 ft.	3 ft.	5 ft.	7 ft.	8 ft.

the curve will not be a straight line. It will be a line as shown in Fig. 27. In such a case where the acceleration is variable, an average value for the rate of change of velocity can be obtained for the whole change so that it being assumed to be constant throughout the period of consideration produces the same change in the velocity. The average acceleration throughout the period is found out by dividing the total change of velocity by the total time taken for the change. In this case it is $(25 \div 5)$, *i.e.*, 5 feet per sec. per sec. This average acceleration produces in 5 seconds a change in the velocity by an amount (5×5), *i.e.*, 25 feet per second changing the velocity from 5 feet per second to 30 feet per second. It is to be marked here that in case of uniformly varying velocity the average acceleration is the same as the constant acceleration.

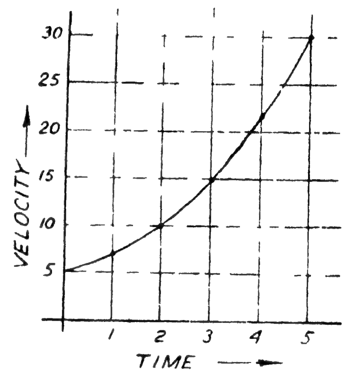


FIG. 27

27. True Acceleration. In the case of a variable acceleration the true acceleration at any instant is the average or mean acceleration during an indefinitely small period of time including that instant.

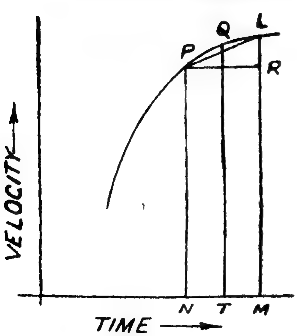


FIG. 28

If the velocity-time curve be drawn and is found to be as shown in Fig. 28, the acceleration can be found out just in the same way as the velocity was obtained from the displacement-time curve (Art. 13). The acceleration at any instant, say, at the

end of the period OT, can be represented by the clinure of the tangent line at Q. The true acceleration at the point T is equal to the mean acceleration which is represented by $\frac{ML-MR}{NM} = \frac{LR}{PR} = \tan LPR$, during a small interval NM which contains the point T. The angle LPR is equal to the angle which the tangent line at Q makes with the horizontal axis.

In the form of Calculus, $f = \frac{dv}{dt}$, where $\frac{dv}{dt}$ represents the slope of the curve at an instant.

$$\text{Hence, } f = \frac{dv}{dt} = \frac{d}{dt} \cdot \frac{ds}{dt} = \frac{d^2s}{dt^2}.$$

Illus. Ex. 9. *From the velocity and time curve in Illus. Ex. 1 find out the accelerations at different instants up to the 60th second from the starting point and draw the acceleration-time curve, representing acceleration by the vertical axis and time by the horizontal axis. Read from the curve the acceleration at the 48th second.*

From the curve (Fig. 7) the velocities at the 5th and 15th seconds are found to be 9.5 and 20 feet respectively.

$\therefore$ the acc. at the 10th. sec. will be $(20 - 9.5)/10 = 1.05$ ft. per sec. per sec.

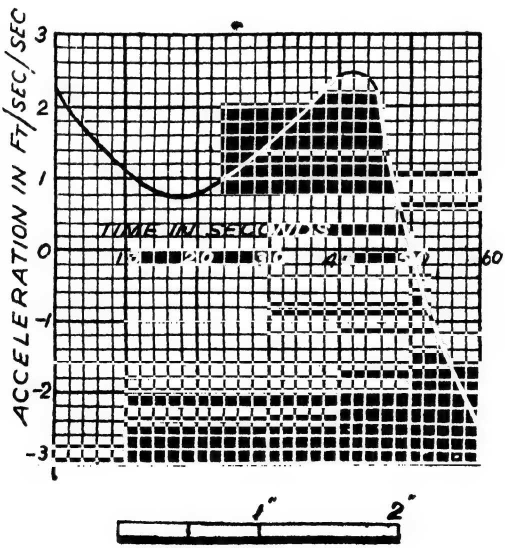


FIG. 29

In the same way,

the acc at the 20th sec will be	(28 — 20)/10 =	8	ft per sec per sec
30th	(41 5 — 28)/10 =	1 35	ft
40th	(65 — 41 5)/10 =	2 45	ft
50th	(67 — 65)/10 =	2	ft
60th	(45 — 67)/10 =	—2 2	ft

Attempting to draw the curve from the above few data it is found to be very difficult to develop the actual curve So the accelerations at a few other points, say, at the 5th, 15th, 45th and 55th seconds are found out in the same way They are 1 6, 8, 2 2 and — 1 4 feet per sec per sec respectively Now tabulate the results as follows,

t	5	10	15	20	30	40	45	50	55	60
f	1 5	1 05	8	8	1 35	2 45	2 2	2	— 1 4	— 2 2

Taking the scale of the curve as one side of a big square in the graph paper to represent vertically 1 foot and horizontally 10 seconds, the curve is drawn as shown in the diagram (Fig 29)

At the 48th second it is found that the height of the curve at that point is 1 2 times the side of a big square Therefore, the acceleration is 1 2 ft per sec per sec and because the value is positive the direction is the same with the velocity

Alternative Method

The alternative method is to draw the differential curve from the velocity-time curve as shown in the diagram below (Fig 30)

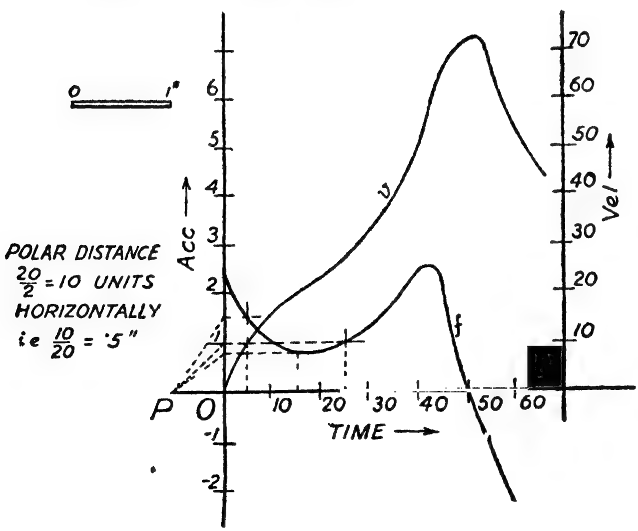


FIG. 30

Illus. Ex. 10. *From the acceleration-time curve obtained in the previous example draw the velocity-time curve and space-time curve respectively.*

From the acceleration-time curve obtained in the previous example draw the velocity-time curve and space-time curve respectively.

The integral curve derived from the acceleration-time curve is the velocity-time curve and the integral curve from that velocity-time curve is the space-time curve.

The ordinary method of drawing the velocity-time curve from the acceleration-time curve is as shown below.

From the acceleration-time curve,

Acceleration at the	5th. second	=	1.5	ft. per sec. per sec.
..	15th. second	=	0.8	ft. ..
..	25th. second	=	0.95	ft. ..
..	35th. second	=	1.9	ft. .. ..
..	45th. second	=	2.2	ft. .. ..
..	55th. second	=	−1.4	ft. .. ..

Acceleration at the 5th. second = 1.5 ft.,
$\therefore$ velocity at the 10th sec. $= 1.5 \times 10 = 15'/$ sec.

Acceleration at the 15th. second = 0.8 ft.,
$\therefore$ velocity at the 20th. sec. $= (0.8 \times 10) + 15 = 23'/$sec.

Acceleration at the 25th. second = 0.95 ft.,
$\therefore$ velocity at the 30th. sec. $= (0.95 \times 10) + 23 = 32.5'/$sec.

Acceleration at the 35th. second = 1.9 ft.,
$\therefore$ velocity at the 40th. sec. $= (1.9 \times 10) + 32.5 = 51.5'/$sec.

Acceleration at the 45th. second = 2.2 ft.,
$\therefore$ velocity at the 50th. sec. $= (2.2 \times 10) + 51.5 = 73.5'/$sec

Acceleration at the 55th. second = − 1.4 ft.,
$\therefore$ velocity at the 60th. sec. $= (-1.4 \times 10) + 73.5 = 59.5'/$sec

From these data obtained, and choosing the scale for velocity, say 1″ along the vertical axis to represent 20 feet, draw the velocity-time curve.

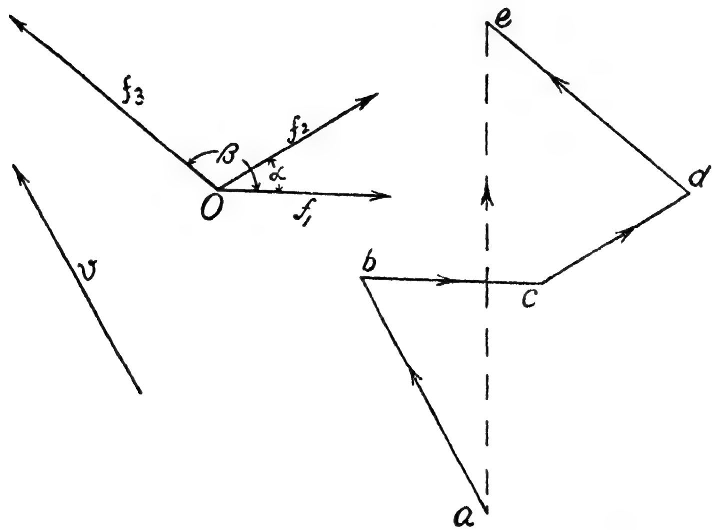
It is to be noted that the velocity and acceleration curves can be drawn against space base as well, *i.e.*, instead of taking the horizontal axis as a time axis, it is taken as a space axis, and they are called velocity-space curve and acceleration-space curve respectively.

28. Resultant and Component Accelerations. As acceleration is a vector quantity the operations of the vector diagrams are utilised in finding out the resultant acceleration in the same way as the resultant velocity was obtained. The composition and resolution of acceleration will follow the same principles as in the case of velocities.

29. ***To determine the velocity after unit time of a particle moving with uniform speed if more than one simultaneous accelerations are introduced in it.***

Let v be the velocity of the particle and f_1, f_2, f_3, are the accelerations produced simultaneously in (Fig. 31)

Construction : Take a point a, from a draw a straight line ab to represent the velocity v in magnitude and direction (Fig. 31 b). From b draw bc to represent f_1 in magnitude and direction, from c draw cd and from d draw de to represent f_2 and f_3 respectively in magnitude and direction. Join ae, then, ae represents in magnitude and direction the velocity after unit time. It is to be marked that it is nothing but the Vector Addition, velocity and acceleration both being vector quantities.



(a) Fig. 31 (b)

30. Relations amongst u, v, f, s and t, where

u = initial velocity
v = final velocity
f = acceleration (constant)
s = displacement
and t = the period of time for consideration.

I. $\frac{dv}{dt} = f$, or, $dv = f.\,dt$,

$\int dv = \int f.\,dt$, if f is constant,

$v = ft + c$, when $t = 0$, *i.e.*, at the beginning of the period, $c = v$, *i.e.*, the constant of integration is equal to the initial velocity which is represented by u.

$\therefore \quad v = u + ft$

Or, it may be proved as follows:

after	1 second	the velocity	will	be	$u + f$
..	2 seconds		..	..	$u + 2f$
..	3 ..	..	..	..	$u + 3f$
..	..	..	..	..	..
..	..	..	..	..	..
..	t ..	..	..	..	$u + tf$

That is, in case of a uniformly accelerated velocity, $v = u + ft$. In case of velocity with uniform retardation, therefore, $v = u - ft$.

Illus. Ex. 11. *The velocity of a train running with a speed of 30 feet per second, is accelerated by 2 feet per second per second. What will be the velocity of the train after 20 seconds?*

$u = 30$ ft. per second	$v = u + ft$,
$f = 2$ ft. per sec. per sec.	Substituting the values of u, f and t,
$t = 20$ secs.	$v = 30 + 2 \times 20 = 70$ feet per second.

Mark, there are four quantities in the equation. If any three of them are known, the fourth unknown quantity can easily be solved out.

II. $v = \frac{ds}{dt} = u + ft$

or, $ds = (u + ft)\,dt$

$\therefore \quad \int ds = \int (u + ft)\,dt$

$\therefore \quad s = ut + \frac{1}{2} ft^2$, constant of integration becomes zero, when $t = 0$.

Or, it may be solved as follows:

It has already been said that in case of uniformly varying velocity, the displacement is equal to the product of the mean velocity and the time,

$$s = \frac{u+v}{2}\, t \qquad \text{(Art. 15, case II—}Eq.\ 1\text{)}$$

But $v = u + ft$. Substituting the value of v in the equation

for $s = \frac{u+u+ft}{2}\, t = ut + \frac{1}{2}\, ft^2$.

In case of uniform retardation the form will be.

$$s = ut - \tfrac{1}{2}ft^2.$$

Here also it is found that there are four quantities s, u, t and f in the equation. Therefore, if any three of them are known, the fourth unknown quantity can easily be solved out.

Illus. Ex. 12. *The velocity of a motor car changes from 30 feet per second to 45 feet per second in 40 seconds. Find out the distance travelled by the car in the time.*

$u = 30$ ft./sec.
$v = 45$ ft./sec.
$t = 40$ secs.
$s = ?$

Here, it is found that the relation (II) cannot be directly applied to find the value of s. First the value of f is found out from the data given and then substituting the value of u, t and f in the relation (II) the value of s is easily found out.

Now, from the relation (I), $v = u + ft$, substituting the values of u, v and t, $45 = 30 + 40f$ or, $f = .375$ feet per second per second. Next, substituting the values of u, t and f in the relation (II),

$$s = 30 \times 40 + \tfrac{1}{2} \times .375 \times (40)^2 = 1500 \text{ feet.}$$

Or directly, $s = \frac{u+v}{2}\, t = \frac{30+45}{2} \times 40 = 1500$ feet.

III. $v = u + ft$.

Squaring both the sides of the equation

$$v^2 = u^2 + 2u.ft + f^2t^2 = u^2 + 2.f\,(ut + \tfrac{1}{2}ft^2) = u^2 + 2fs$$

In case of uniform retardation, the form of the equation will be $v^2 = u^2 - 2fs$.

Here again the equation has four quantities v, u, f and s. If any three of them are known the fourth unknown quantity can be easily found out. It is to be marked that in case III where the body starts from rest, *i.e.*, when u is zero, $v = \sqrt{2fs}$.

Illus. Ex. 13. *While describing a straight path of 2000 feet the velocity of a motor cycle changes from 30 feet per second to 60 feet per second. What is the acceleration?*

$u = 30$ ft./sec.
$v = 60$ ft./sec.
$s = 2000$ ft.
$f = \ ?$

The relation (III) can be directly utilised to solve the problem. Thus, substituting the values of v, u and s in the equation,

$(60)^2 = (30)^2 + 2 \times f \times 2000$, or, $f = .675$ ft. per sec. per sec.

Thus the three equations are,

$$1.\quad v = u \pm ft \qquad \cdots\cdots Eq.\ 2$$
$$2.\quad s = ut \pm \tfrac{1}{2} ft^2 \qquad \cdots\cdots Eq.\ 3$$
$$3.\quad v^2 = u^2 \pm 2\ ft \qquad \cdots\cdots Eq.\ 4$$

From the forms of the equations it is evident that in case of a motion with constant acceleration, the acceleration-time curve is a straight line parallel to the time-axis (X-axis), the velocity-time curve is a straight line inclined to the time-axis cutting the velocity-axis at a point depending on the value of the initial velocity, and the space-time curve is a parabola. It is to be marked that the velocity-space curve is also a parabola.

Illus. Ex. 14. *In travelling for 45 minutes a train starts from rest at a station A and comes to rest again at another station B. At a definite place C between A and B the train attains the highest velocity 45 miles per hour. If the acceleration and the retardation of the train be uniform and equal in magnitudes, find the distance between A and B.*

In this case u and v being equal to zero, and the acceleration and retardation being equal in magnitude, the highest velocity will be attained at the midway between A and B. Therefore, the distance between A and B will be equal to twice the distance travelled during the accelerating period or the retarding period, *i.e.*, if s be the distance

$$s = 2.\ (\tfrac{1}{2}.\ f.\ t^2),\ u \text{ being equal to zero.}$$

$$= 2.\ \tfrac{1}{2}.\ f.\ \left(\frac{45}{2}\times 60\right)^2. \quad \text{But } v = ft. \text{ where } u = 0$$

$$\therefore f = \frac{66}{22.5\times 60} \text{ ft. per sec. per sec.}$$

$$\therefore s = 2.\ \tfrac{1}{2}.\ \frac{66}{22.5\times 60}(22.5 \times 60)^2 \text{ feet.}$$

$$= 16.87 \text{ miles.}$$

INSTANCE OF MOTION WITH CONSTANT ACCELERATION

MOTION UNDER GRAVITY

31. It is found on experiment that when a particle is projected up or down in any direction whatsoever there is always an acceleration in the particle in the vertically downward direction. The magnitude of this acceleration is constant for a definite place on the surface of the earth. Symbolically it is represented by the letter 'g' and is known as the *acceleration due to gravity*. It is approximately equal to 981 centimetres or 32.2 feet per second per second in London. The problems of this book are solved with this value. The value of g varies slightly at different places. It is the greatest at the poles and least at the equator. All about it will be discussed in details in the next chapter.

Particles moving under gravity may be classified generally in the following three different groups :

I. Particles thrown up or down along the vertical direction,

II. Particles thrown up or down in any other direction (projectiles),

III. Particles moving up or down a smooth inclined plane.

32. *Group I.* In cases of particles moving vertically up or down the relations amongst the initial velocity, final velocity, acceleration, the height moved and the time of travel can be put as follows (representing the height by the letter h), just in the same way as the equations explained in Article 30.

1. $v = u \pm gt$ ....*Eq.* 5
2. $h = ut \pm \frac{1}{2}gt^2$ ....*Eq.* 6
3. $v^2 = u^2 \pm 2gh$ ....*Eq.* 7

Velocity and acceleration in the downward direction are generally taken as positive.

In cases of dropped particles where $u = 0$, $v = \sqrt{2gh}$ or $h = \dfrac{v^2}{2g}$. Again if a particle is projected upwards vertically with an initial velocity u_1 and if it can rise a height h_1, then, v being

equal to zero, $u_1 = \sqrt{2gh_1}$, or, $h_1 = u_1^2/2g$. Therefore, h must be equal to h_1, when $u_1 = v$.

33. *Conventions of Positive and Negative Signs.* *In solving problems the convention adopted in this treatise is that the motion in the upward direction and in a direction from left to right with respect to the point of reference are taken as positive and otherwise as negative.* Therefore, the displacement must be taken in the same sense with the motion.

Illus. Ex. 15. *A ball is thrown upwards with a velocity of 150 feet per second in a direction at right angles to the surface of the earth. Another ball is projected along the same path after an interval of 3 seconds with the same velocity. When and where will the two balls collide?*

If the first ball collides with the second after t seconds, the time required by the second ball is $(t - 3)$ seconds. Again, because the two balls will meet at the same height,

$$h = 150\,t - \tfrac{1}{2} \times 32.2 \times t^2 = 150\,(t - 3) - \tfrac{1}{2} \times 32.2 \times (t - 3)^2$$

From which t is found to be 6.15 seconds, and substituting this value of t in one of the relations, h is found to be 312.75 feet.

Illus. Ex. 16. *A ball being dropped from the top of a tower 100 metres high meets another ball which was thrown upwards vertically from the ground at the same instant at the midway. Find the velocity of projection of the second ball.*

If t and h be the time and height, when and where the balls meet, then h being the height of the midway,

$$h = \tfrac{1}{2}\,g\,t^2 = u_1 t - \tfrac{1}{2}\,g\,t^2,$$ where u_1 is the velocity of the projection of the 2nd ball

$$or,\ 50 = \tfrac{1}{2} \times 9.81 \times t^2, \text{ or, } t^2 = \frac{100}{9.81}$$

$$\text{Again, } 50 = u_1 \sqrt{\frac{100}{9.81}} - \tfrac{1}{2} \times 9.81 \times \frac{100}{9.81}$$

$$\therefore \quad u_1 = 31.32 \text{ metres per second.}$$

Illus. Ex. 17. *From the top of a tower a person projects a ball vertically upwards with a velocity of 29.43 metres per second. 4 seconds later he drops another ball which reaches the ground simultaneously with the first one. Find the height of the tower and the time of fall for the second ball.*

Let h be the height of the tower.

Then, $h = 29.43\,t - \tfrac{1}{2} \times 9.81 \times t^2 = -\tfrac{1}{2} \times 9.81 \times (t - 4)^2$

From which, $t = 8$ seconds.

Substituting this value of t in the first relation of h

$$h = 78.48 \text{ metres.}$$

Group II.—PROJECTILES.

34. There are particles which do not move in a straight line though start with straight motion. A projectile, *i.e.*, a particle thrown with an initial straight motion in a direction other than the vertical, does not describe a straight path. From actual experiments it is found that the path of a projectile is a parabola and this result can also be mathematically derived.

From experiments it is found that if from a height a particle is dropped and simultaneously another particle is thrown horizontally with any speed whatsoever, the two particles will reach the ground at the same instant. From this fact it can easily be concluded that the horizontal motion of the second particle has no effect on its downward vertical motion.

With regard to the direction of projection, projectiles may be divided into three different sections :

I. Particles thrown making some angle up the horizontal direction,

II. Particles thrown horizontally from a higher level,

III. Particles thrown from a higher level making some angle down the horizontal direction.

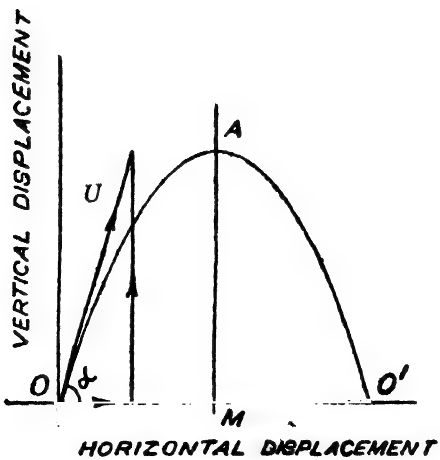


FIG. 32

35. Before considering the motion of these particles the terms of different quantities that arise during the discussions on the motion of projectiles and their explanations should be studied first.

1. *Point of Projection.* The point from which the projectile is thrown is known as the point of projection. O is the point of projection (Fig. 32).

2. *Velocity of Projection.* The initial velocity with which a particle is thrown is called the velocity of projection—u is the velocity of projection (Fig. 32).

3. *Angle of Projection.* The angle between the direction of the velocity of projection and the horizontal direction is called the angle of projection. The angle α is the angle of projection (Fig. 32).

4. *Trajectory.* The path described by the particle is called the trajectory. OAO' is the trajectory (Fig. 32).

5. *Range.* When thrown up the distance between the point of projection and the point where the particle meets again with the horizontal level containing the point of projection is called the range of the projectile along the horizontal direction. OO' is the horizontal range (Fig. 32).

6. *Time of Flight.* The total time required by the particle to describe the path OAO' is called the time of flight.

36. Path of a Projectile is a Parabola. Suppose a particle is thrown up from the point O with a velocity u (Fig. 33). The velocity u can be resolved into two components in the horizontal and the vertical directions respectively as $u \cos \alpha$ and $u \sin \alpha$, where α is the angle of projection. The acceleration due to gravity has no effect on the horizontal component of the velocity, because it is at right angles to the component. Thus, though the horizontal component of the velocity remains constant the vertical component changes every second by an amount g which is always in the vertically downward direction. If v be the velocity at any instant, then $v = \sqrt{v_x^2 + v_y}$, where v_x and v_y are the horizontal and the verticle components respectively.

Take any point P on the trajectory (Fig. 33) and let x and y be the co-ordinates of the particle at the position P and let t be the time to travel the path OP.

Now, $v_x = \dfrac{dx}{dt} = u \cos \alpha$ (1)

$v_y = \dfrac{dy}{dt} = u \sin \alpha - gt$ (2)

Integrating (1) & (2),

$x = u \cos \alpha . t$ (3)

$y = u \sin \alpha . t - \frac{1}{2} g t^2$ (4)

Substituting the value of t from (3) in (4)

$$y = x \tan \alpha - \frac{g x^2}{2 u^2 \cos^2 \alpha} \qquad \textit{Eq. 8}$$

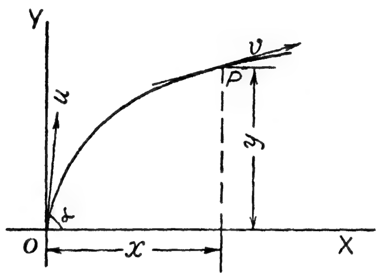


FIG 33

Or, it can be proved as follows

$v_x = u \cos \alpha$, and $v_y = u \sin \alpha - gt$,

$x = u \cos \alpha\, t$ (I)

Therefore, $t = \dfrac{x}{u \cos \alpha}$

And $y = u \sin \alpha\, t - \frac{1}{2} g t^2$ (II)

Substituting the value of t obtained from (I) in (II),

$$y = u \sin \alpha . \frac{x}{u \cos \alpha} - \frac{1}{2} g \frac{x^2}{u^2 \cos^2 \alpha}$$

$$= x . \tan \alpha - x^2 . \frac{g}{2 u^2 \cos^2 \alpha}$$

Thus it takes the form of an equation for a parabola. Therefore, the path of travel of a projectile is a parabola. This is true for all the particles in the three sections.

Illus. Ex. 18. *A body is thrown at an angle of 60° with the horizontal with a velocity of 88 feet per second. Find the maximum height of the flight.*

The vertical component of the velocity of projection is 88 sin 60 = 88 × .866 = 76.2 feet per second and the horizontal component is 88 cos 60 = 44 feet per second.

As the two components are independent of each other, in this case the consideration of the horizontal component is not necessary.

Now, according to the *Eq.* 7, $v^2 = u^2 - 2gh$, where h represents the maximum height, v being equal to zero, $u^2 = 2gh$ or $h = \frac{u^2}{2g}$. Here $h = \frac{u^2 \sin^2 \alpha}{2g}$

Substituting the value of u, α and g

$$h = \frac{(76.2)^2}{64.4} = 90.18 \text{ feet.}$$

Illus. Ex. 19. *In the same problem find the time in which it reaches the maximum height.*

If t be the required time, then according to the *Eq.* 4, $v = u - gt$, v being zero, $u = gt$.

In this case $u \sin \alpha = gt$, or $t = \frac{u \sin \alpha}{g} = \frac{76.2}{32.2} = 2.367$ secs.

Illus. Ex. 20. *In the same problem find the velocity of the body after the 1st. second of the flight. What is the height of the position of the body?*

Let v be the required velocity, which makes an angle θ with the horizontal.

Then, $v = \sqrt{v_x^2 + v_y^2}$. But $v_x = u \cos 60 = 44'/\text{sec.}$

and $v_y = u \sin 60 - g.1.$

$= 76.2 - 32.2 = 34'/\text{sec}$

$\therefore v = \sqrt{44^2 + 34^2} = \sqrt{3092} = 55.6$ ft. per sec.

and $\theta = \tan^{-1} \frac{34}{44} = \tan^{-1} .7727.$

$\therefore \theta = 37.7°$

Illus. Ex. 21. *In the same problem find the velocity of the body when it is at a height of 50 feet from the point of projection.*

If v be the required velocity, $v = \sqrt{v_x^2 + v_y^2}$

But $v_x = u \cos \alpha$, and according to the *Eq.* 7, $v^2 = u^2 - 2gh$

or $v = \sqrt{u^2 - 2gh}$, here $v_y = \sqrt{u^2 \sin^2 \alpha - 2gh}$

$\therefore$ the required velocity $v = \sqrt{u^2 \cos^2 \alpha + u^2 \sin^2 \alpha - 2gh}$

$$= \sqrt{u^2 - 2gh}$$

$$= \sqrt{(88)^2 - 2 \times 32.2 \times 50}$$

$$= 67.25 \text{ ft./sec.}$$

$$\theta = \tan^{-1} \frac{\sqrt{u^2 \sin^2 \alpha - 2gh}}{u \cos \alpha} = \tan^{-1} \frac{\sqrt{(76.2)^2 - 2 \times 32.2 \times 50}}{44}$$

$= \tan^{-1}$ 1.156. That is, $\theta = 49.15°$

Illus. Ex. 22. *What is the time required for the total flight of the body in the same problem?*

The total time must be equal to the sum of the time required to reach the maximum height and the time required to reach the ground. But the rising and falling of the body depend on the vertical component of the velocity of projection and the acceleration due to gravity, and as they are of constant magnitudes, the period of rising and falling must be equal. Let T be the total time. Then,

$T = 2t = 2 \times 2.367 = 4.734$ secs. (*Illus. Ex.* 19)

Illus. Ex. 23. *What is the horizontal range of the same body?*

The horizontal range (OO' — Fig. 32) $= u \cos \alpha \times T$

$$= 44 \times 4.734 = 208.4 \text{ ft.}$$

Illus. Ex. 24. *Find the angle of projection which gives a body its maximum horizontal range with a definite velocity of projection. What is the maximum horizontal range for a body with 88 feet per second as velocity of projection?*

If OO' be the maximum horizontal range in Fig. 32,

$$OO' = u \cos \alpha \times 2t, \text{ but } t = \frac{u \sin \alpha}{g}$$

$$OO' = u \cos \alpha \times \frac{2u \sin \alpha}{g} = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 \sin 2\alpha}{g}$$

Now, OO' is maximum, when sin 2α is maximum, *i.e.*, when

$2\alpha = 90°$, or, $\alpha = 45°$

$\therefore$ the maximum horizontal range for the body with the specified magnitude of the velocity of projection,

$$= \frac{u^2 \sin 2\alpha}{g} = \frac{u^2 \sin 90}{g} = \frac{u^2}{g} = \frac{(88)^2}{32.2} = 240.4 \text{ feet.}$$

Illus. Ex. 25. *Find the angle of projection of a body whose horizontal range 200 feet and the velocity of projection is 88 feet per second.*

8

Horizontal range $= \frac{u^2 \sin 2\alpha}{g} = \frac{u^2 \sin(180-2\alpha)}{g} = \frac{u^2 \sin 2(90-\alpha)}{g}$

Hence, substituting the known values,

$$200 = \frac{(88)^2 \sin 2(90-\alpha)}{32.2}, \text{ from which } \sin 2(90 - 2\alpha) = \frac{6440}{7744}$$

$\therefore\ 2(90 - \alpha) = 56.3°$, or, $\alpha = \frac{180 - 56.3}{2} = 61.85°$

Again, $200 = \frac{(88)^2 \sin 2\alpha}{32.2}$, from which, $2\alpha = 56.3°$

$\therefore\ \alpha = 28.15°$

That is, there are two angles of projection for a definite range and a definite velocity of projection.

It is to be marked that the two angles are equally inclined on either side of the direction of the velocity of projection which gives the maximum horizontal range, because,

$$45 - 28.15 = 16.85°$$
$$\text{and} \quad 61.85 - 45 = 16.85°$$

Illus. Ex. 26. *In the illustrated example 18 find the range of the body on a plane of 30° inclination.*

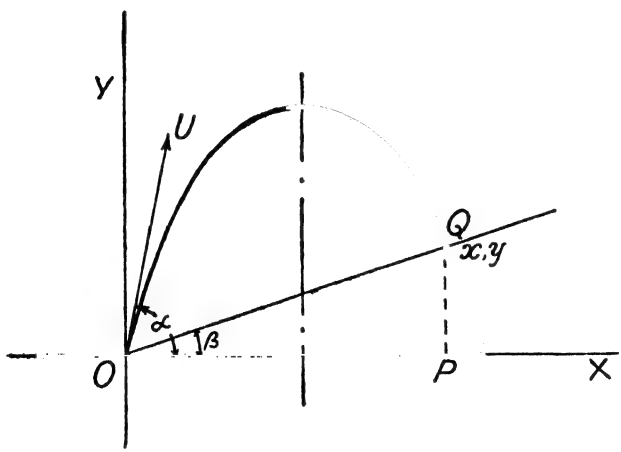


FIG. 34

Let u make an angle α (Fig. 34) with the horizontal plane, and the inclined plane OQ make an angle β with that plane.

Here $\alpha = 60°$ and $\beta = 30°$

Let x, y be the co-ordinates of the point Q, the range on the inclined plane being OQ.

$$y = x \tan \alpha - \frac{g\, x^2}{2\, u^2 \cos^2 \alpha}, \quad \text{but } \frac{y}{x} = \tan 30$$

$$\therefore \quad x = \frac{y}{\tan 30}$$

$$\text{Hence, } y = \frac{y \tan 60}{\tan 30} - \frac{32\ 2\, y^2}{2.\,(88)^2 \cos^2 60 \tan^2 30}$$

$$= 3\ y - .025\ y^2$$

or, $2 = .025\ y$. Therefore, $y = 80$ feet.

Hence, OQ, the range on the inclined plane,

$$= \frac{80}{\sin 30} = 160 \text{ feet.}$$

Illus. Ex. 27. *A stone is thrown up from a stiff cliff 161 feet high with a velocity of 322 feet per second in a direction making an angle of 30° with the horizon. Find, when and where it will strike the ground.*

The vertical component of the velocity $= 322 \sin 30$, or 161 feet per second and the horizontal component $= 322 \cos 30$, or 278.9 feet per second.

The horizontal component has no effect on the vertical motion. The stone will continue to go up till the vertical component becomes zero and the time required for it is $\frac{161}{g}$ secs. Then it will begin to fall and in time $\frac{161}{g}$ secs. again it will come to the level of the point of projection ; where its downward vertical component of the velocity is 161 feet per second. With this initial velocity and acceleration g downwards, the stone will travel 161 feet. Therefore, according to the equation, $h = ut + \frac{1}{2} g t^2$, $161 = 161\, t + \frac{1}{2} \times 32.2\, t^2$ or, $t = .9$ sec. Therefore, the total time $= \left(2 \times \frac{161}{g}\right) + .9 = 10.9$ secs.

Again, the horizontal component of the velocity of the stone is 278.9 feet per sec. Therefore, in 10.9 secs. the body will travel horizontally a distance 10.9×278.9 or about 3040 feet from the point of projection before it strikes the ground.

Illus. Ex. 28. *In a competition of 'throwing a cricket ball' the record shows that the winning boy threw the ball 115 metres, which was his maximum. Find the velocity of projection. What is the maximum height of the flight? What is the magnitude and direction of the velocity of the ball at a height of 20 metres? What is the total time of flight?*

The range is the greatest when the angle of projection is 45° and the value is given by $\frac{u^2}{g}$.

Hence $\frac{u^2}{g} = 115$ or $u^2 = 115 \times 9.81$,

i.e., $u = 33.6$ metres per sec. approximately.

Again, the maximum height and the total time are given by the relations,

$$h = \frac{(u^2 \sin^2 \alpha)}{2g} \text{ and } t = \frac{(2u \sin \alpha)}{g}.$$

The height will be maximum when the angle of projection is 45°, because the range is maximum at that angle.

Therefore, the maximum height $= \frac{(33.6)^2 \times \sin^2 45}{2 \times 9.81} = 28.8$ metres.

Time $= (2 \times 33.6 \times .7071) \div 9.81 = 4.84$ seconds.

The velocity at the height of 20 metres, according to the equation,

$v = \sqrt{u^2 - 2gh}$, is equal to $\sqrt{(33.6)^2 - (2 \times 9.81 \times 20)}$

$= 26.68$ metres per second.

Now, the horizontal component remaining constant throughout the period, $v \cos \theta = u \cos \alpha$, where v and θ are the velocity and the angle of flight at a definite height. Therefore, in this case, $26.68 \cos \theta = 33.6 \cos 45$, or $\cos \theta = .8825$

$\therefore \theta = 28°$ approximately.

37. When a projectile is thrown with an initial velocity u, from a top horizontally (α, the angle of projection being zero) the form of the equation of the locus will be, $y = -\frac{g}{2u^2} x^2$, where the time is calculated from the origin O. The ($-$) sign indicates that the projectile is falling downwards, because in section 1, y in the upward direction was taken in the positive sense. It is an equation for a parabola, the vertex of which is the point of projection as the vertical component of the velocity of projection there is zero.

Illus. Ex. 29. *A fruit is projected horizontally from the top of a tree 80 feet high. If the focus of the parabolic path described by the fruit be in the horizontal plane through the foot of the tree, find the velocity of projection.*

$$\frac{u^2}{2g} = 80. \qquad \therefore u^2 = 2 \times 32.2 \times 80$$

$$u = 71.78 \text{ feet per second.}$$

38. In case where the projectile is thrown from a top making some angle α downwards with the horizontal direction having an initial velocity u.

If t be the time to reach a point, then the horizontal component of the displacement from the vertical line through the point of projection

$x = u \cos \alpha\, t$ and the vertical component from the level of the point of projection,

$$y = -(u \sin \alpha t + \tfrac{1}{2} g t^2)$$

Now, substituting the value of t from, $x = u \cos \alpha\, t$ the magnitude

of $y = \tan \alpha . x + \frac{g}{2u^2 \cos^2 \alpha} \cdot x^2.$

It is the form of an equation of a parabola The vertex of the path will be at a height where $u \sin \alpha$ becomes zero due to the acceleration of gravity. Therefore, the height will be equal to $\frac{u^2 \sin^2 \alpha}{2g}$, which must be higher than the point of projection. The axis will be at a distance of $\frac{u \sin \alpha . u \cos \alpha}{g}$, *ie.*, $\frac{u^2 \sin \alpha \cos \alpha}{g}$ away from the vertical line through the point of projection, because the time required to reach the vertex from the point of projection if the velocity be u in the direction as shown in the diagram is $\frac{u \sin \alpha}{g}$.

Illus. Ex. 30. *A tennis ball is served from a height of 7.5 feet. The ball just passes over the centre of the net by 1 in. and hits the service line. If the horizontal distances of the server and the service line be 39 and 21 feet respectively from the net and if the central height of the net be 3 feet, find the horizontal component of the velocity of projection of the ball and also the angle of projection.*

If t_1 and t_2 be the time to reach the net and the service line respectively and if u be the velocity of projection,

$$u \cos \alpha\ t_1 = 39, \text{ or, } t_1 = \frac{39}{u \cos \alpha}$$

$$\text{Again, } u \cos \alpha\ t_2 = 60, \text{ or, } t_2 = \frac{60}{u \cos \alpha}$$

$$\text{but } u \sin \alpha\ t_1 + \tfrac{1}{2} g t_1^2 = \frac{53}{12}$$

$$\text{and } u \sin \alpha\ t_2 + \tfrac{1}{2} g t_2^2 = 7.5$$

Now, substituting the value of t_1 and t_2

$$39 \tan \alpha + \tfrac{1}{2} g \left(\frac{39}{u \cos \alpha} \right)^2 = \frac{53}{12}$$

$$\text{and } 60 \tan \alpha + \tfrac{1}{2}g\left(\frac{60}{u \cos \alpha}\right)^2 = \frac{15}{2}$$

$$\text{or} \qquad \tan \alpha + \tfrac{1}{2}g\frac{39}{u^2 \cos^2 \alpha} = \frac{53}{12 \times 39}$$

$$\text{and} \qquad \tan \alpha + \tfrac{1}{2}g\frac{60}{u^2 \cos^2 \alpha} = \frac{15}{2 \times 60}.$$

Now, subtracting,

$$\tfrac{1}{2}g\frac{1}{u^2 \cos^2 \alpha}(60-39) = \frac{15}{2 \times 60} - \frac{53}{12 \times 39}, \text{ from which,}$$

$$u \cos \alpha = \sqrt{\frac{21 \times 32\ 2 \times 2 \times 60 \times 39}{2 \times 55}} = 169\ 6 \text{ feet per second.}$$

Substituting the value of $u \cos \alpha$ in any of the second equations, $\tan \alpha = 0914$, therefore, $\alpha = 5° - 1'$

Group III.—MOTION ON A SMOOTH INCLINED PLANE.

39. If a particle be placed on a smooth inclined plane and is allowed to move freely the particle moves down along the plane.

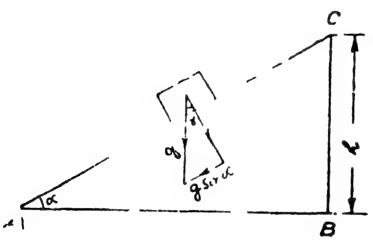


FIG 35

Suppose the inclination of a plane is α degrees with the horizontal plane (Fig. 35). The particle starts from rest at C, to find its velocity when it comes at A. Let BC be equal to h.

It has already been said that every particle gains an acceleration g vertically downwards if there is nothing to obstruct. In this case, due to the presence of the plane AC, the particle cannot move in the vertically downward direction. It will move along the plane down with an acceleration equal to the component of the acceleration g along the plane.

The acceleration g can be divided into two components—one at right angles to the plane and the other along the plane downwards. The first one, $g \cos \alpha$, has no effect on the motion of the particle being at right angles to the direction of motion. The second one, $g \sin \alpha$, creates the motion of the particle down the plane. Under the condition, therefore, the velocity of the particle at A will be equal to $\sqrt{2.g \sin \alpha . AC}$, but $AC = h \div \sin \alpha$.

Therefore the velocity

$$= \sqrt{2\, g \sin \alpha \,.\, \frac{h}{\sin \alpha}} = \sqrt{2 \,.\, g \,.\, h}$$

just as if the particle falls from the height h. Thus, the velocity of a moving particle at any point on an inclined plane depends on the height of the point from which it takes the start.

Illus. Ex. 31. *Find the velocity of a motor car when it runs 1000 feet down a plane 1 in 200 without the help of the engine.*

$\sin \alpha = 1/200$	If s be the distance travelled,
$g \quad = 32.2$ ft. per sec. per sec.	then,
$s \quad = 1000$ ft.	$s = h/\sin \alpha$
$u \quad = 0$	or, $1000 = h \div \frac{1}{200}$
$v \quad = ?$	$\therefore \quad h = 1000 \div 200 = 5$ ft.

Now, $v = \sqrt{2.g.h}$. $\therefore$ substituting the value of g and h in the equation,

$$v = \sqrt{2 \times 32.2 \times 5} = 17.9 \text{ ft. per second.}$$

40. If the particle has got an initial velocity u at C, the velocity of the particle at A, $v = \sqrt{u^2 + 2g \sin \alpha .\ AC} = \sqrt{u^2 + 2gh}$.

When the particle goes up the inclined plane, the velocity of the particle decreases by an amount $g \sin \alpha$ per unit of time. Therefore, if the velocity of the particle at A be u' and if it changes to v' when it reaches the point C, $v' = \sqrt{u'^2 - 2g \sin \alpha .\ AC} = \sqrt{u'^2 - 2gh}$.

Illus. Ex. 32. *A driver ceases the action of an automobile engine when it reaches the foot of an inclined track with a velocity of 60 miles per hour. Find how much the velocity of the car will be reduced when it rises 1000 feet up the track if the angle of inclination be 5 degrees.*

$$h = 1000 \sin 5 = 1000 \times .0872 = 87.2 \text{ feet.}$$

$$\therefore v' = \sqrt{88^2 - 2 \times 32.2 \times 87.2} = 46.18 \text{ feet per second.}$$

$$= 46.18 \times \frac{15}{22} = 31.5 \text{ miles per hour.}$$

$\therefore$ the reduction of velocity $= (60 - 31.5) = 28.5$ miles per hour.

41. Application of the Principle of Motion on a Smooth Inclined Plane.

1. *Motion down a smooth curve path along a concave surface.*

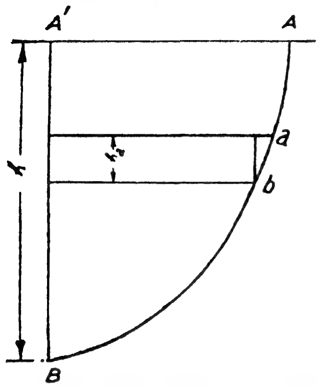


FIG. 36

Let AB be the smooth curved path (Fig. 36)—A being at a height of h from B. Take any two points a and b very close to each other on the path. Because these two points are very near to each other, ab, though a portion of the curved path, may be taken approximately to be an ordinary inclined plane. If, v_a and v_b be the velocities of the particle at a and b respectively, $v_b^2 = v_a^2 + 2gh_a$, where h_a is the height of a from b. Hence, $v_b^2 - v_a^2 = 2gh_a$.

Now if the path be divided into n number of small segments and if similar differences between the squares of the two velocities for all the segments be added,

$$v_2^2 - v_1^2 = 2gh_1 + 2gh_2 + 2gh_3 + \cdots + 2gh_a + 2gh_b + \cdots + 2gh_n$$
$$= 2g\,(h_1 + h_2 + h_3 + \cdots + h_a + h_b + \cdots + h_n)$$
$$= 2gh,$$ where v_2 and v_1 are the velocities of the particles at B and A respectively and h_1, h_2, h_3, etc. are the vertical depths of the segments respectively. Hence, $v^2{}_2 = v_1^2 + 2gh$.

If the body starts from rest, *i.e.*, if v_1 be zero, $v_2 = \sqrt{2gh}$. Like Art. 39, in this case too it is found that the velocity of a particle starting from rest at A becomes at B equal to the velocity that the particle would develop if it were allowed to fall freely from the height h. Similarly, if v_2 be the velocity of a particle at B in going up to A, the velocity becomes such that $v_1^2 = v_2^2 - 2gh$.

Now, since AB is taken to be any curve the foregoing results will be true for each and every regular curve.

Illus. Ex. 33. *A cyclist reaches the foot of a vertical smooth curved path of radius 50 ft. with a velocity of 15 miles per hour and stops paddling. Find how far the cyclist will be able to go up the path before coming to rest if the straight horizontal path be tangent to the curved path.* (Fig. 37)

15 miles per hour = 22 ft. per sec. if h be the vertical height that the cyclist can reach due to his motion,

$$(22)^2 = \sqrt{2 \times 32.2 \times h}$$

$$\therefore \quad h = 7.51 \text{ ft.}$$

Now from the diagram it is found that if r be the radius of the curved path,

$$(r - h) = r \cos \theta, \text{ or } \cos \theta = 1 - \frac{h}{r}$$

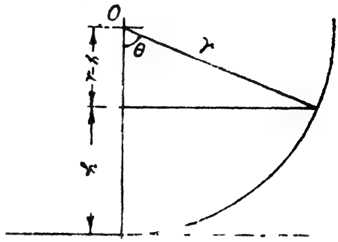


FIG. 37

Therefore, in this case $\cos \theta = [1 - (7.51 \div 50)] = (1 - .1502) = .8498$

$$\therefore \theta = 32^\circ \text{ approximately.}$$

$\therefore$ the length of the path along the curved path travelled by the cyclist $= 32 \times 2\pi r/360 = 27.9$ feet.

2. *Motion down along a chord in a vertical circle having its one end at the highest point.*

Suppose a particle is to travel from A, the highest point in a vertical circle starting from rest, along a chord of the circle (Fig. 38). Let AB' be a chord other than the vertical diameter AB. The inclination of the chord with the horizontal direction is $(90 - \theta)$, where θ is the angle made by the chord with the vertical diameter. The length of the chord is, in terms of r (radius), equal to $2r \cos \theta$, and hence the height of the point A from the lowest point of the chord is $2 . r . \cos^2 \theta$. The component of the acceleration g along the chord is $g . \cos \theta$.

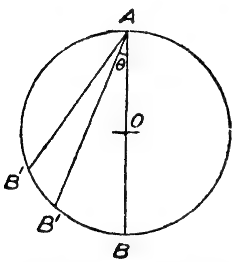


FIG. 38

Therefore, the velocity of the particle at B', $v_b' = \sqrt{2 . g . 2r \cos^2 \theta} = 2 . \cos \theta \sqrt{gr}$, and the time t to travel the distance AB' is equal to

$$\frac{2 \cos \theta \sqrt{g\pi}}{g \cos \theta} = 2 \sqrt{\frac{\pi}{g}}$$

Thus it is found that t is independent of θ and the length of the path of travel. It is also to be noticed that the velocity of a particle at the foot of a chord depends on the value of θ.

3. *The path of travel of a particle from a given point to a given circle, all in the same vertical plane, so that the time required will be the least one.*

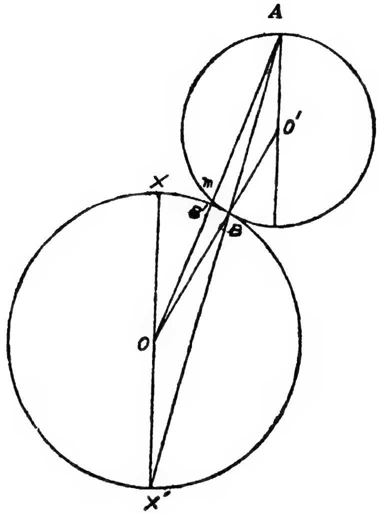


FIG. 39

Suppose A is the given point and XBX' is the given circle, whose centre is O (Fig. 39). Through A draw a vertical line and through O draw a diameter XX' parallel to it. Join AX' cutting the circle at B. AB is the path for the quickest descent. Produce the radius OB to O' to meet the vertical line through A. With centre O' and radius $O'A$ or $O'B$ draw a circle. Join AO cutting the given circle and the circle drawn at B' and m respectively. Now, according to the previous case the time to travel from A to B, and from A to m is the same. But AB' is greater than Am, therefore, the time to travel the distance AB' is greater than that to travel Am. Again, AB' is the shortest distance between the point A and the given circle. Thus, the time to travel along any other line is greater than the time that is required to travel along AB. Hence, AB is the required path along which the particle may reach the given circle within the least period of time, *i.e.*, AB is the path for the quickest descent.

Illus. Ex. 34. *The horizontal distance between the lower and the upper end of an inclined plane remaining constant, what should be the inclination so that a body may slide down in the least time starting from rest at the top?*

Let α be the inclination. If $AB = l$ (constant),

$$AC = \frac{l}{\cos \alpha} \quad \text{(Fig. 35)}$$

$$= \tfrac{1}{2} f t^2$$

But $f = g \sin \alpha$

$$\therefore \quad \frac{l}{\cos \alpha} = \tfrac{1}{2} g \sin \alpha \; t^2$$

$$\text{or } t^2 = \frac{2l}{g \sin \alpha \cos \alpha}$$

$$\therefore \quad t = \sqrt{\frac{2l}{g \sin \alpha \cos \alpha}}$$

Now t is least when ($\sin \alpha \cos \alpha$) is maximum, *i.e.*, when $\frac{\sin 2\alpha}{2}$ is maximum, *i.e.*, when $\sin 2\alpha$ is maximum.

Now $\sin 2\alpha$ is maximum, when $2\alpha = 90°$, *i.e.*, when $\alpha = 45°$

INSTANCES OF MOTION WITH VARYING ACCELERATION

Curvilinear Motion and Simple Harmonic Motion

42. Curvilinear Motion. Motion, not in a straight line, is a *Curvilinear Motion.* In this sense the motion of a projectile is also a curvilinear motion, because the path of motion of a projectile is a parabola. However, the case was treated along with other instances of motions with constant acceleration. Mark, that the acceleration that was discussed was the linear acceleration and its magnitude and direction were constant.

Here, the instances of motion with varying acceleration, of course linear, will be treated, but it is to be remembered that the motions are not straight motions.

Because the motion is along a curved path, any two positions on the path may be represented by an angular displacement with reference to a definite point. Before proceeding with the instances of motion with variable acceleration, some discussion about angular motion will be made.

43. Varying Acceleration. Acceleration is said to be *varying* when its magnitude or direction or both undergo a change.

44. Angular Motion. If a particle instead of moving in a straight line rotates about a definite point, or, a definite axis, in a plane at right angles to the axis, the particle is said to have *angular motion* about that point or axis. *If at an instant a particle is found to occupy a position P_1 (Fig. 40) and at a subsequent instant, while rotating about a definite* point O, it is found to occupy another position P_2, then the angle described by the particle at the point O is $P_1 O P_2$, which is the angular displacement of the particle about the point O. For the same positions of P_1 and P_2 the particle will have a different angular displacement $P_1 O_1 P_2$ about another point O_1. Thus, the angular displacement of a particle depends on the point about which it rotates. The angular displacement is generally measured in radians. When the case is referred to axes, O and O_1 denote the traces of the axes.

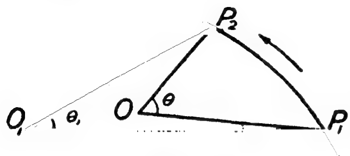


FIG. 40

45. Angular Velocity. The *angular velocity* of a moving particle about a definite point is the time rate of angular displacement about that point. The angular velocity is usually represented in radians per second and is denoted by the Greek alphabet ω. The angular velocity just like linear velocity, may be uniform as well as variable.

46. Uniform Angular Velocity. A particle is said to have *uniform angular velocity* about a definite point when it describes equal amount of angle in each unit of time about that point, otherwise it is said to move with varying angular velocity.

47. Average Angular Velocity. In case of varying angular velocity the *average angular velocity* is found out by dividing the total angular displacement by the total time taken.

48. True Angular Velocity. In case of varying angular velocity the *true angular velocity* at any instant is the average angular velocity during a very small interval of time including that instant. $\omega = \dfrac{d\,\theta}{d\,t}$ where $d\,\theta$ is a small increment in angular displacement in a small time dt.

49. Angular Acceleration. The *angular acceleration* is the rate of change of angular velocity. It is usually represented by angles in

radians per second per second and is generally denoted by the Greek alphabet α. The acceleration also may be uniform as well as variable. In case of variable acceleration the true acceleration at any instant is the mean acceleration during an infinitely small period of time including that instant.

$$\alpha = \frac{d\,\omega}{d\,t} = \frac{d^2\,\theta}{d\,t^2}$$

Following the same procedure as was done in case of straight line motion the following three equations establishing the relations between the angular velocity, angular acceleration, angular displacement and time may be formed :

1. $\omega = \omega_0 \pm \alpha\, t$ ······ *Eq.* 9
2. $\theta = \omega_0 t \pm \frac{1}{2}\, \alpha\, t^2$ ······*Eq.* 10
3. $\omega^2 = \omega_0^2 \pm 2\, \alpha\, \theta$ ······*Eq.* 11

where ω is the final velocity during a period of time t and ω_0 is the initial velocity and α & θ are the angular acceleration and displacement respectively.

50. Circular Motion. When a particle in motion describes a circular path about a definite point, the particle is said to possess *circular motion* about that point. If the angular or the linear speed of the particle be uniform the motion is called uniform circular motion.

51. Direction of Linear Motion of a particle moving in a circular path at any instant. About the direction of motion of a body Newton's first law of motion (which will be found in the next chapter) states—if there is a motion in a body in a definite direction then that body will continue to move in that direction unless some effort is made to change that direction.

Thus, when a body moves in a circular path, at any instant the direction of linear motion is tangential to the path of rotation at that instant. Because it is evident that when two consecutive points on the path at any instant is joined and produced both ways it will be a straight line at right angles to the radius of the circular path at that instant.

But, this direction of motion does not remain constant as some effort is made to change the direction to move the body in a circular path.

52. Relation between the Linear and Angular Motion of a particle rotating in a circle with constant speed.

Let v be the linear speed and ω be the corresponding angular velocity of a particle which moves in a circle uniformly about a definite point O (Fig. 41) with radius r. If t be the time required to move from P_1 to P_2 and if the angle $P_1O\,P_2$ be θ radians, $\omega = \dfrac{\theta}{t}$, but $\theta =$ (arc) $P_1\,P_2 \div r$, and (arc) $P_1\,P_2 = v.\,t$.

Therefore, $\omega = \dfrac{v.\,t}{r} \div t = \dfrac{v}{r}$.

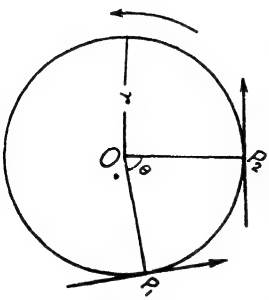


FIG. 41

This is always true when the relative linear speed of the particle with respect to the point O remains constant as v. It is immaterial whether the point O has got a motion or not. For example, take the case of a street roller. While in motion its axle has got a horizontal motion, whereas the rim has got a horizontal as well as circular motion about the axle. If the relative linear velocity of the particles in the rim with respect to the points in the axis of the axle about which they are rotating respectively be v (rim thickness being neglected), then $\omega = \dfrac{v}{r}$, where r is the radius of the rolling body.

Thus, the magnitude of the angular velocity is equal to the magnitude of the linear speed divided by the radius of the circle in which the particle is rotating.

It is obvious that the binding of this relation between the linear and angular motions of a body rotating in a circular path is not only applicable to uniform motion but also to accelerated motion the same relation is maintained. In the latter case, at any instant, if v be the linear velocity and ω be the angular velocity of a body rotating in a circular path of radius r, $v = \omega.r$ at that instant.

53. Acceleration of a particle moving in a circular path with accelerated motion. In time t the particle moves from A to P (Fig. 42) describing an angle θ at the centre. From P drop a

perpendicular PM on OX. Then, the component of the position along OX is x. But $x = r \cos \theta$, where r is the radius of the circular path. Therefore, the velocity along OX,

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(r \cos \theta)$$

$$= -r \sin \theta \frac{d\theta}{dt}$$

where v_x is the component of the velocity of P along OX at that instant.

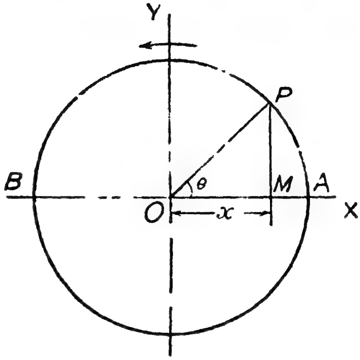


FIG. 42

And the acceleration, f_x, along

$$OX = \frac{d}{dt}\left(-r \sin \theta \frac{d\theta}{dt}\right)$$

$$= -r \cos \left(\frac{d\theta}{dt}\right)^2 - r \sin \theta \frac{d^2\theta}{dt^2}$$

When $\theta = O$, f_x becomes radial and is equal to, $-r\left(\frac{d\theta}{dt}\right)^2$

$$= -\omega^2 r = -\frac{v^2}{r} = -\omega v \qquad \cdots\cdots\cdots\cdots\cdots Eq.\ 12$$

Again, the velocity along OY, *i.e.*, the transverse velocity,

$$v_y = \frac{dy}{dt}, \text{ but } y = r \sin \theta$$

$$\text{Therefore, } v_y = \frac{d}{dt}(r \sin \theta) = r \cos \theta . \frac{d\theta}{dt}$$

Hence, the acceleration along OY, *i.e.*, transverse acceleration,

$$f_T = \frac{d}{dt}\left(r \cos \theta . \frac{d\theta}{dt}\right)$$

$$= -r \sin \theta \left(\frac{d\theta}{dt}\right)^2 + r \cos \theta \frac{d^2\theta}{dt^2}$$

When $\theta = O$,

f_T becomes tangential and is equal to,

$$r\frac{d^2\theta}{dt^2} = r.\alpha = f \qquad \cdots\cdots\cdots Eq.\ 13$$

where f is the linear acceleration.

The negative and the positive signs before the two accelerations above indicate the directions—the former towards the centre and the latter in the same direction with the velocity.

The problem may be treated geometrically in the following way :

Suppose, a particle is moving in a circular path of radius r, in a direction as shown by the arrow-head (Fig. 43). Let v_1 be its linear

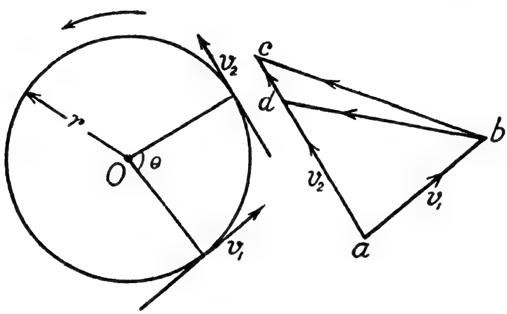


FIG. 43

velocity at a definite instant and also let f be its linear acceleration and v_2 be its velocity after a time t, during which it covers an angular displacement of θ radians.

Draw the vector diagram abc to determine the change in velocity during this time. The vectors ab and ac represent the velocities v_1 and v_2 respectively in magnitudes and directions. Then, bc is the vector subtraction and represents the change in velocity. Take a point d on ac so that $ad = ab$. Join bd. From the vector diagram, $ac - ab = bc$, and, $bd + dc = bc$. Thus, the change in velocity, bc, can be resolved into two components, bd and dc.

Therefore, acceleration can also be resolved into two components along bd and dc. Hence, $\frac{bc}{t} = \frac{bd}{t} + \frac{dc}{t}$ (Geometric sum).

When θ is very small, $bd = v_1.\theta$, and $t = \frac{\theta}{\frac{1}{2}(\omega_1 + \omega_2)}$, where ω_1 and ω_2 are the two corresponding angular velocities for v_1 and v_2 respectively.

But, $\omega_1 = \frac{v_1}{}$ and $\omega_2 = \frac{v_2}{}$

Therefore, $t = \dfrac{2\,\theta\, r}{(v_1 + v_2)}$

Hence, $\dfrac{bd}{t} = \dfrac{v_1\,\theta}{\dfrac{2\,\theta\,r}{(v_1+v_2)}} = \dfrac{v_1\,(v_1+v_2)}{2r}$

But, when θ is very small,

$v_1 + v_2 = 2\,v_1$ very approximately.

$\therefore \quad \dfrac{bd}{t} = \dfrac{v_1^{\,2}}{r} = \omega^2\, r = \omega\, v_1$, or when put in general forms, acceleration along bd,

$= \dfrac{v^2}{r} = \omega^2 r = \omega v$, where v is the velocity at an instant.

Again, $\dfrac{dc}{t} = \dfrac{ac-ad}{t} = \dfrac{v_2-v_1}{t} = \dfrac{f.\,t}{t} = f.$

Therefore, the acceleration of the particle along bd, $= \dfrac{v_1^{\,2}}{r}$ (or $\omega^2 r$ or ωv_1) $+ f$. Now, if θ be gradually reduced, then, at the limit, bd becomes perpendicular to ab, which represents v_1 in magnitude and direction. Hence, the direction of the component acceleration, $\dfrac{v_1^{\,2}}{r}$ is at right angles to the direction of v_1 at that instant, *i.e.*, towards the centre of the circular path as is evident from the construction of the diagram. This component acceleration is called the radial acceleration and is generally represented by f_r (f_r of the previous method). Again, in the limiting value of θ, when bd is perpendicular to ab, ac becomes perpendicular to bd. Hence, the direction of this component acceleration is along the direction of the velocity v_1, *i.e.*, tangential to the path of rotation, and it is generally represented by f_T.

Thus, when a particle moves in a circular path with an accelerated motion, there are two simultaneous accelerations—one is radial to an amount equal to $\dfrac{v^2}{r}$ or $\dfrac{\omega^2}{r}$ or $\omega\, v$, and the other tangential to an amount equal to the linear acceleration of the particle. In general form v is used in place of v_1.

54. Condition of Uniform Circular Motion of a particle. From the previous article it is clear that when a particle moves in a circular path with uniform motion, the vector diagram

(*Fig.* 43) reduces to a form *abd*, because in that case $dc = 0$. Therefore, only one acceleration remains and that is the radial one. Thus, when a particle moves in a circle with uniform motion, there is an acceleration by an amount equal to $\frac{v^2}{r}$ or $\omega^2 r$ or ωv towards the centre of the circular path ; or in other words, to rotate a body in a circular path with uniform motion an acceleration to an amount equal to $\frac{\omega^2}{r}$ or $\omega^2 r$ or ωv must be created towards the centre of the circular path.

In case of uniform circular motion the magnitude of the acceleration remains constant but the direction changes every moment. Whereas, in case of uniformly accelerated circular motion the tangential acceleration remains constant in magnitude but varies in direction at every instant and the radial acceleration changes both in magnitude and direction at every instant.

Illus. Ex. 35. *The crank shaft and the wheel of a cycle is so geared that for one complete revolution of the pedal, the wheel rotates once. If the length of the crank be 10 in. and the diameter of the wheel be 100 in., find the angular and linear velocity of the pedal centre when the cycle is running at 15 miles per hour. What is the axial acceleration produced in the pedal?*

The velocity ratio between the pedal centre and the wheel is

$$(10 \times 2) \div 100 = \tfrac{1}{5}.$$

$\therefore$ the linear velocity of the pedal centre is $(15 \div 5) = 3$ miles per hour $= 4.4$ ft. per sec.

$\therefore$ the angular velocity $= \frac{4.4 \times 12}{10} = 5.28$ radians per sec.

$\therefore$ the acceleration produced towards the axle $= (5.28)^2 \times \frac{10}{12}$

$= 23.77$ ft. per sec. per sec.

55. Plane Curve Motion. The term 'plane curve motion' or simply 'plane motion' means in general all the curvilinear motions, when the motions of composing particles of a body are parallel to a common plane. The term is really applicable to rigid bodies. Here, we shall treat the general motion of only one particle—not moving in a path having a constant value for 'r', *i.e.*, the case where r changes at every instant with the motion—in a single plane.

Following the notations of the foregoing article,

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(r\cos\theta) = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}$$

And $$f_x = \frac{d}{dt}(v_x) = \frac{d}{dt}\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right)$$

$$= \frac{d^2r}{dt^2}\cos\theta - \sin\theta.\frac{d\theta}{dt}.\frac{dr}{dt} - r\sin\theta\frac{d^2\theta}{dt^2} -$$

$$r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - \sin\theta.\frac{dr}{dt}.\frac{d\theta}{dt}$$

$$= \frac{d^2r}{dt^2}\cos\theta - 2\sin\theta\frac{dr}{d\theta}.\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 -$$

$$r\sin\theta\frac{d^2\theta}{dt^2}$$

When $\theta = O$,

f_x becomes radial and is equal to, $\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$

Putting $\dfrac{dr}{dt} = \rho$

$$f_x = \frac{d\rho}{dt} - r\,\omega^2 \qquad \text{..........................} Eq.\ 14$$

Again $$v_y = \frac{dy}{dt} = \frac{d}{dt}r\sin\theta = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$$

$$\therefore\quad f_T = \frac{d}{dt}\left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)$$

$$= \frac{d^2r}{dt^2}\sin\theta + \cos\theta\frac{dr}{dt}.\frac{d\theta}{dt} + r\cos\theta\frac{d^2\theta}{dt^2} +$$

$$\cos\theta\frac{dr}{dt}.\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{d^2r}{dt^2}\sin\theta + 2\cos\theta\frac{dr}{dt}.\frac{d\theta}{dt} + r\cos\theta\frac{d^2\theta}{dt^2} -$$

$$r\sin\theta\left(\frac{d\theta}{dt}\right)^2$$

When $\theta = O$, $$f_r = \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$= 2\,\rho\omega + r\frac{d\omega}{dt} \qquad \cdots\cdots\cdots Eq.\ 15$$

When $\rho = O$, the result becomes the same as in article 53. The factor $2\rho\omega$ is called the 'Coriolis' acceleration. The effect of the change of the radius is to change both the radial and the transverse accelerations—the former by an amount, $d\rho/dt$, and the latter by the amount (coriolis component) $2\rho\omega$.

SIMPLE HARMONIC MOTION

This is a type of motion which can neither be said straight nor curvilinear. Though the motion is along a straight path it should not and cannot be said to be a straight motion as it will be evident from the nature of the motion described below.

56. When a particle rotates in a circle with uniform linear velocity it is found that if the velocities at successive points on the path be resolved into components along two rectangular co-ordinate axes passing through the centre of the circular path, the component velocities along the axes change following a definite law. The motion abiding by this law is called the *simple harmonic motion.*

It is found that this motion has got an acceleration proportional to the distance along the axis concerned from the centre and is always directed towards it. Thus, a simple harmonic motion may be defined as follows:

When a particle moves to and fro on either side of a point in a straight line so that its acceleration is proportional to its distance along the path from that point and is always directed towards it, the particle is said to move in a simple harmonic motion. Symbolically the conditions may be represented by the form, $f = -k.x$, where k is a constant and f and x are the acceleration and the distance respectively. The negative sign indicates that where x is positive, the acceleration is negative and *vice versa.* Whether x on the right hand side of the point will be taken as positive or otherwise is immaterial. Generally x is measured positive on the right hand side. The physical significance of the constant k is that it is the magnitude of the acceleration at unit distance.

57. The point about which the particle moves is called the *mid-position* or the *neutral position*. The maximum displacement of the particle from the mid-position is called the *amplitude*.* One complete *oscillation* is the swing from one extreme end of the path to the other and back again. The swing from one extreme end to the other is called a *beat*.

If a particle moves on either side of the point O (Fig. 44) along AB and BA under the conditions mentioned above, O is called the mid-position. OA or OB is the amplitude. $AB + BA$ is the oscillation. AB or BA is a beat.

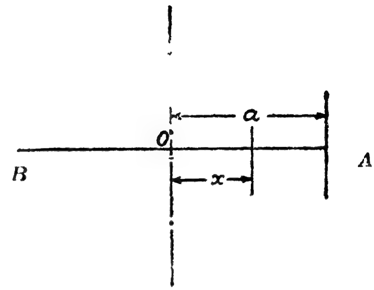


FIG. 44

58. The approximate examples of this kind of motion are found in cases of spring vibrations, pendulum oscillations, piston motions of engines, etc. In cases of springs and pistons, the motions are in a straight line whereas the motion of the bob of a pendulum is in a curved path ; but both kinds of motion obey approximately the conditions of simple harmonic motion. This motion is nothing but a type of reciprocating motion.

59. Sir William Thomson (Lord Kelvin) and P. G. Tait defined simple harmonic motion as follows:

When a point Q moves uniformly in a circle the perpendicular QP drawn from its position at any instant to a fixed diameter AB of the circle intersects the diameter at a point P whose position changes by a simple harmonic motion. (Fig. 45)

Now, $f = \dfrac{d^2x}{dt^2} = -kx$ (Art. 56)

Multiplying both the sides by $2\dfrac{dx}{dt}dt$,

---

* Really the term 'amplitude' means the maximum value of a variable quantity. Here, with respect to the definite point (mid-position) the maximum linear displacement is OA. Therefore, OA is the amplitude with respect to the variable quantity, x, the linear displacement.

$$2\frac{dx}{dt}\,dt.\frac{d^2x}{dt^2} = -2\frac{dx}{dt}\,dt.\ k\,x$$

$$\text{or, } 2.d\frac{dx}{dt}\cdot\frac{dx}{dt} = -2k \times dx$$

By integration $\left(\frac{dx}{dt}\right)^2 = -k\,x^2 + C_1$, but $\frac{dx}{dt} = 0$,

when $x = a$ $\therefore\ C_1 = k\,a^2$

Therefore, $\left(\frac{dx}{dt}\right)^2 = k\,a^2 - k\,x^2$

Hence, $\frac{dx}{dt} = \sqrt{k\,a^2 - k\,x^2}$

or, $dt = \frac{dx}{\sqrt{k\,a^2 - k\,x^2}}$

Again by integration,

$$t = \frac{1}{\sqrt{k}}\sin^{-1}\frac{\sqrt{k}.\ x}{\sqrt{k}.\ a} + C_2 = \frac{1}{\sqrt{k}}\sin^{-1}\frac{x}{a} + C_2$$

when $t = O$, $x = a$

and C_2 becomes equal to $-\frac{1}{\sqrt{k}}\sin^{-1}1 = \frac{\pi}{2\sqrt{k}}$

Hence, $t = \frac{1}{\sqrt{k}}\sin^{-1}\frac{x}{a} - \frac{1}{\sqrt{k}}\sin^{-1}1$

or, $\sqrt{k}.t = \sin^{-1}\frac{x}{a} - \sin^{-1}1$

or, $\sqrt{k}.t + \frac{\pi}{2} = \sin^{-1}\frac{x}{a}$, *i.e.*, $\sin\left(\sqrt{k}.t + \frac{\pi}{2}\right) = \frac{x}{a}$

$\therefore$ $x = a\sin(90 + \sqrt{k}.t) = a\cos\sqrt{k}.t$

$$v = \frac{dx}{dt} = -a\sqrt{k}\sin\sqrt{k}.\,t$$

and $f = \frac{d^2x}{dt^2} = -ak\cos\sqrt{k}.\,t$

$= -kx$, that is, $x = a\cos\sqrt{k}.\,t$

But, when a body rotates in a circle with a constant angular velocity ω, the angular displacement θ in time $t = \omega t$, and $x = a \cos \theta = a \cos \omega t$. Therefore, $\sqrt{k} = \omega$, or $k = \omega^2$.

It has already been noted that in equation $f = -kx$, $k = f$ when $x = 1$, *i.e.*, the constant k represents the magnitude of the acceleration at unit distance, which is again equal to a value ω^2, ω being the angular velocity.

Or

It can be treated as follows.

Suppose a particle Q is moving in a circle of radius a in a direction as shown in the diagram (Fig. 45) with a uniform linear speed v. Take a position of Q on the circular path so that the radius OQ makes an angle θ radians with the diameter AOB. Take any other position of the particle, say, Q_1 which is very near to Q and let Q_1O make an angle θ_1 radians with the radius OA. Let P and P_1 be the projections of the positions Q and Q_1 respectively on the diameter AOB. Let v_1 and v_2 be the respective linear velocities of P and P_1 along the diameter. Also let OP and OP_1 be measured as x and x_1 respectively. It is evident that the velocity v of the particle at the position Q can be resolved into two components—in a direction parallel to the diameter AOB and in a direction at right angles to it. The consideration of the component which is at right angles to the diameter having no effect on the motion of the projectional point along the diameter is rejected. The component along the diameter, which is equal to $v \sin \theta$ is the velocity of P at a distance x from the mid-position. Thus,

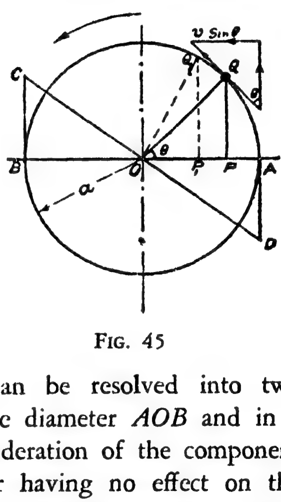


FIG. 45

$v_1 = v \sin \theta = v. \dfrac{QP}{QO}$. But in the triangle QOP, $OP = x$ and $OQ = a$. Therefore, $QP = \pm \sqrt{a^2 - x^2}$. Hence substituting the values of QP and QO, $v_1 = \pm v \dfrac{\sqrt{a^2 - x^2}}{a} = \pm \dfrac{v}{a} \sqrt{a^2 - x^2}$.

Again, $x = a \cos \theta$ and $\omega = \dfrac{v}{a}$, where ω is the angular velocity of

the particle about O. Therefore, $v_1 = \pm \omega\sqrt{a^2(1 - \cos^2\theta)} = \pm a\omega \sin\theta$. But $\theta = \omega t$, where t is the time required by the particle to move from A to Q.

Therefore, $v_1 = \pm a\omega \sin \omega t$.

Similarly $v_2 = \pm \frac{v}{a}\sqrt{a^2 - x_1^2} = \pm a\omega \sin \omega t_1$, where t_1 is the time to describe the angle θ_1. These are the velocities of P and P_1 with reference to their distances from the mid-position. Thus, for any definite value of x the velocity of the point P may have both positive and negative value. If the time be measured from A it is evident that with respect to time, the velocity must have a single value $- a\omega \sin \omega t$ (direction—left to right is taken as positive and right to left as negative).

Now, the mean velocity between P and P_1 is equal to $\frac{1}{2}(v_1 + v_2)$ and, therefore, the time required to travel the distance is,

$$PP_1 \div \tfrac{1}{2}(v_1 + v_2) = \frac{x_1 - x}{\frac{1}{2}(v_1 + v_2)} = \frac{2(x_1 - x)}{v_1 + v_2}.$$

Hence, if f be the mean acceleration of the point during this time,

$$f = (v_2 - v_1) \div \frac{2(x_1 - x)}{v_1 + v_2} = \frac{v_2^2 - v_1^2}{2(x_1 - x)}.$$

Now, substituting the values of v_1 and v_2,

$$f = \frac{v^2(x^2 - x_1^2)}{a^2 . 2(x_1 - x)}$$

$$= -\tfrac{1}{2} . \frac{v^2}{a^2}(x + x_1) = -\tfrac{1}{2}\omega^2(x + x_1).$$

If P_1 be taken very near to P, x and x_1 are approximately equal. Therefore, f becomes equal to $-\frac{v^2}{a^2}x$ or $-\omega^2 x$ and this is the true acceleration of the point P at the distance x from the mid-position. Now, if ω^2 is represented by a single letter k, the form of the equation stands as $f = -kx$. Thus, when a particle Q moves uniformly in a circle about a definite point O with radius a, the point of projection P of the position Q moves along the diameter AOB satisfying all the conditions of simple harmonic motion and with reference to time,

Velocity $v = -a\omega \sin \omega t$ or $-a\sqrt{k} \sin \sqrt{k}\, t$ ....*Eq.* 16

Distance from the mid-position,

$$x = a \cos \omega t \text{ or } a \cos \sqrt{k}\, t \qquad \text{.........} Eq.\ 17$$

Acceleration,

$$f = -k x = -a \omega^2 \cos \omega t = -a k \cos \sqrt{k}\, t \qquad \text{......} Eq.\ 18$$

60. From the equations, 16, 17 and 18 it is found that if the velocity-time curve, the acceleration-time curve and the space-time curve be drawn, the first one will be a sine curve and the rest two will be cosine curves respectively as shown in Figure 46. The vertical axis represents the variable quantities and the horizontal axis represents the angular displacement. It is to be mentioned here that the horizontal axis may also be chosen to represent time and the curves of the same nature may be drawn to show the change in the variables with the time.

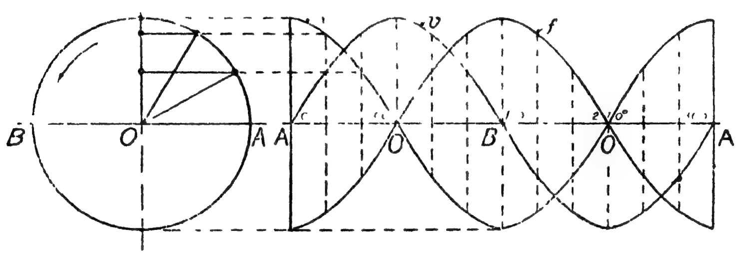


FIG. 46

61. But if the velocity and acceleration curves be drawn against an axis representing linear displacement (space) they will not be sine and cosine curves respectively as mentioned above.

In the diagram (Fig. 45) if *AOB* be taken to represent displacement and the vertical line through *B* represents acceleration in a definite scale, the space-acceleration curve will be a straight line of the nature of *COD*, because the equation of the acceleration is an equation of a straight line, $f = -k x$.

The equation for the velocity at any point is

$$v = \pm \sqrt{k}.\ \sqrt{a^2 - x^2}$$

From which $v^2 = k\ (a^2 - x^2)$ or $kx^2 + v^2 = ka^2$ and this is the form of an equation of an ellipse. Therefore, the space-velocity curve is an ellipse. But again, $QP = a \sin\theta$ and $v_1 = v \sin\theta$. Therefore, if the scale of the velocity be taken in such a way that a represents v, then QP will represent v_1 on the same scale and the circular path itself will represent the velocity diagram against the space base AOB.

62. From the above discussions it is evident that velocity is maximum at the mid-position when $x = 0$, *i.e.*, when $\theta = 90°$ & $270°$. Therefore, the amplitude of the velocity is represented by the ordinates of the velocity curve at 90° & 270°. Similarly, acceleration being maximum when $x = a$, *i.e.*, when $\theta = 0°$ & $180°$, the ordinates of the acceleration curve at 0° & 180° represent the amplitude of the acceleration. The velocity and acceleration are equal to zero at $\theta = 0°$ & $180°$ and $\theta = 90°$ & $270°$ respectively, *i.e.*, when $x = a$ and $x = 0$ respectively.

63. If t be the time for completing the circular path in second,

$$t = \frac{2\pi}{\omega}$$

Again, ω being equal to $\sqrt{k}$, $t = \dfrac{2\pi}{\sqrt{k}}$,

$$\therefore\quad k = \frac{4\pi^2}{t^2} \qquad \cdots\cdots\cdots\cdots Eq.\ 19$$

t, the time for one complete oscillation, *i.e.*, the period for travelling from A to B and back to A again (Fig. 45), is called the *periodic time*.

Equation for the periodic time is,

$$t = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k}} \qquad \cdots\cdots\cdots\cdots Eq.\ 20$$

The reciprocal of the periodic time is the number of complete oscillation per unit time and it is called the *frequency*. Therefore, if n be the frequency,

$$n = \frac{1}{t} = \frac{\omega}{2\pi} \qquad \cdots\cdots\cdots\cdots Eq.\ 21$$

Here, as the time unit has been taken as a second, n is the number of oscillation per second. If N be the frequency per minute,

$$N = \frac{30}{\pi}\omega \qquad \cdots\cdots Eq.\ 22$$

64. **Phase and Epoch angles.** If the time be reckoned from C instead of A (Fig. 47), then, $\theta = \omega t + \varepsilon$. $(\omega t + \varepsilon)$ is called the *phase angle* of the motion, ε is called the *epoch angle* or simply *epoch* of the motion.

When ε is positive (in the first quadrant of the figure) it is called *Lead*, and when it is negative (in the fourth quadrant) it is called *Lag*.

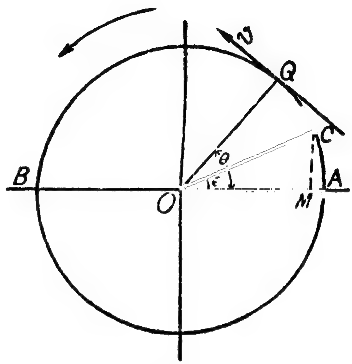


FIG. 47

Now substituting the value of θ in the *Equations* 16, 17 & 18

$$v = -a\omega \sin(\omega t + \varepsilon) \qquad \cdots\cdots Eq.\ 23$$

$$x = a\cos(\omega t + \varepsilon) \qquad \cdots\cdots Eq.\ 24$$

$$\text{and } f = -a\omega^2 \cos(\omega t + \varepsilon) \qquad \cdots\cdots Eq.\ 25$$

65. In an engine the trace of the crank pin centre line in a vertical plane may be taken as the position of the particle discussed above. Therefore, it is clear that the bigger and bigger be the ratio between the connecting rod and the crank lengths, the more and more will the motion of the piston follow the law of simple harmonic motion. Of course, for other reasons which will be clear with the advanced knowledge of the science of machines, a limiting ratio is maintained in the crank and connecting rod mechanism between their lengths. General custom of assuming the ratio is 1 : 5.

66. The actual motion of the piston and the cross-head will not be a simple harmonic motion. It will differ by a very small amount due to the link arrangement to change the motion from circular to motion of translation. However, the difference can be neglected if the lengths of the crank and the connecting rod are properly chosen.

In the diagram (Fig. 48) C and C_1 are the two positions of the cross-head, which has the same motion with the position.

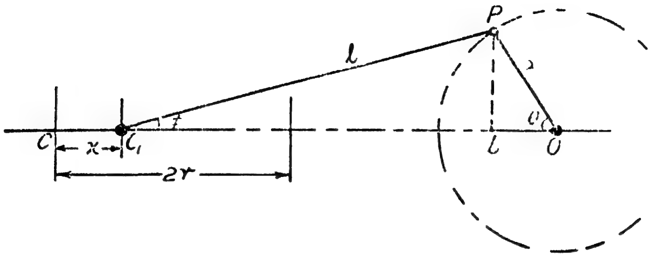


FIG 48

In the diagram, $x = OC - OC_1 = OC - (Ob + bC_1)$

$$= l + r - (r \cos\theta + l \cos\phi)$$

In the triangle, PC_1O, $\dfrac{r}{\sin\phi} = \dfrac{l}{\sin\theta}$, or, $\sin\phi = \dfrac{r}{l}\sin\theta$

$$= \frac{\sin\theta}{n}, \text{ where } n = \frac{l}{r}$$

But $\cos\phi = (1 - \sin^2\phi)^{\frac{1}{2}} = \left(1 - \dfrac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}} = \dfrac{1}{n}(n^2 - \sin^2\theta)^{\frac{1}{2}}$

$$\therefore \quad x = nr + r - \left\{ r\cos\theta + nr\,\frac{1}{n}\left(n^2 - \sin^2\theta\right)^{\frac{1}{2}} \right\}$$

$$= r\left\{ n + 1 - \cos\theta - \left(n^2 - \sin^2\theta\right)^{\frac{1}{2}} \right\}$$

$$\therefore \quad v = \frac{dx}{dt} = r\Bigg[\sin\theta\,\frac{d\theta}{dt} - \left\{ \frac{1}{2}\left(n^2 - \sin^2\theta\right)^{-\frac{1}{2}} \times \frac{d}{d\sin\theta}\left(n^2 - \sin^2\theta\right) \times \frac{d}{d\theta}\sin\theta \times \frac{d\theta}{dt} \right\} \Bigg]$$

$$= r\Bigg[\sin\theta\,\frac{d\theta}{dt} - \left\{ \frac{1}{2}\left(n^2 - \sin^2\theta\right)^{\frac{1}{2}} \times -2\sin\theta \times \cos\theta \times \frac{d\theta}{dt} \right\} \Bigg]$$

$$= r\left[\sin\theta\frac{d\theta}{dt} + \frac{\sin 2\theta}{2\sqrt{n^2-\sin^2\theta}}\cdot\frac{d\theta}{dt}\right]$$

$$= r\cdot\frac{d\theta}{dt}\left[\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2-\sin^2\theta}}\right]$$

$$= r\omega\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \text{ approximately} \qquad Eq.\ 26$$

$n^2 - \sin^2\theta = n^2$ approximately, because θ being generally very small, $\sin^2\theta$ is far too small a quantity and can be neglected when compared with n^2.

Similarly,

$$f = \frac{dv}{dt} = r\omega\left(\frac{d\theta}{dt}\cos\theta + \frac{d\theta}{dt}\cdot\frac{2\cos 2\theta}{2n}\right)$$

$$= r\omega^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right) \qquad Eq.\ 27$$

The values of v and f can be put as follows :

$$v = r\omega\sin\omega t + r\omega\frac{\sin 2\omega t}{2n}$$

$$= r\sqrt{k}\sin\sqrt{k}\,t + r\sqrt{k}\frac{\sin 2\sqrt{k}\,t}{2n} \quad \cdots\cdots \quad Eq.\ 28$$

and $$f = r\omega^2\cos\omega t + r\omega^2\frac{\cos 2\omega t}{n}$$

$$= rk\cos\sqrt{k}\,t + rk\frac{\cos 2\sqrt{k}\,t}{n} \quad \cdots\cdots\cdots \quad Eq.\ 29$$

It is to be marked that in each of the *Equations* 28 and 29 the first term is in the form of S. H. M. With the values two quantities are added respectively which can be neglected when n is a great number.

Illus. Ex. 36. *The periodic time for a part of a steam engine moving in a simple harmonic motion is 1 second. If the amplitude be 10 in., find the magnitude of the velocity of the part at a distance of 3 in. from the mid-position. At what distance from the mid-position the acceleration is 6.573 feet per second per second? What will be the magnitudes of the velocity and the acceleration after .1 second from one extreme end of the amplitude?*

$$t = \frac{2\pi}{\sqrt{k}}, \text{ here } t \text{ being equal to } 1,$$

$$\sqrt{k} = 2\pi = 2 \times 3.14 = 6.28$$

$$\therefore \quad k = 39.45$$

1. $v = \sqrt{k} . \sqrt{a^2 - x^2} = 6.28 \quad \sqrt{10^2 - 3^2} = 6.28 \times 9.539$ in. per sec.

$= (6.28 \times 9.539)/12 = 4.878$ feet per sec.

2. $f = -k.x$, substituting the values of f and k,

$$6.573 = -39.45.x \quad \text{or} \quad x = -\frac{6.573}{39.45} = -\frac{1}{6} \text{ foot} = -2 \text{ in.}$$

i.e., the part is at a distance of 2 in. from the mid-position on the negative side.

3. $v = a \sqrt{k} \sin \sqrt{k}\, t$, substituting the values,

$v = 10 \times 6.28 \sin (6.28 \times .1)^c = 10 \times 6.28 \sin 36° = 36.92$ in.

$= 3.076$ feet per second.

4. $f = a k \cos \sqrt{k}\, t$, substituting the values,

$f = 10 \times 39.45 \times \cos (6.28 \times .1)^c = 10 \times 39.45 \cos 36°$

$= 26.32$ feet per sec. per sec.

PROBLEMS

23. The acceleration-time curve for a definite period of an express train is drawn. With the help of a planimeter the area under the curve is measured and found to be 3.01 sq. inches. The base of the area is 4.3 inches long and represents 21.5 seconds. The acceleration scale is 1 inch to 5 feet per second per second. If the magnitude of the velocity at the beginning be 12.75 feet per second, determine its magnitude at the end of 21.5 seconds in miles per hour. *Ans.* 60 miles per hour.

24. The acceleration-space curve of a moving body is drawn on a base of 3.2 inches, which represents 32 feet. A planimeter gives the area under the curve as 5.25 sq. inches. If the acceleration scale is 1 inch to 5 feet per second per second and if the initial velocity to travel the space be 10 feet per second, find the final velocity. *Ans.* 25 feet per second.

25. The scales in a velocity-time curve are, 1 inch vertical represents 50 feet and 1 inch horizontal represents 10 seconds. At a certain instant the tangent to the curve makes an angle 21.8 degrees with the horizontal. What is the acceleration of the motion at that instant?

Ans. 2 feet per sec. per sec.

26. A train starts from rest and runs with an acceleration of 2.2 feet per second per second for 20 seconds. Find the velocity of the train in miles per hour and also find the distance covered in that time.

Ans. 30 miles per hour, 440 feet.

27. With what initial velocity was a motor car running if the velocity of the car attained a speed of 30 miles per hour in 242 feet when the engine exerted a force to create an acceleration of 2 feet per second per second?

Ans. 21.21 miles per hour.

28. An automobile engine starts from rest and moving with uniform acceleration describes 494 centimetres in the 10th second. Find the acceleration. *Ans.* 52 cms. per sec. per sec.

29. A train starts from rest and moving with uniform acceleration for 20 seconds runs with constant velocity for 5 seconds and covers 750 metres during the time. Find the acceleration. *Ans.* 250 cms. per sec. per sec.

30. Find the acceleration of a train if it passes one station at the rate of 30 miles per hour and another station 1.5 miles apart at the rate of 45 miles per hour. *Ans.* .1528 ft. per sec. per sec.

31. A particle moves with uniform acceleration. In the 10th and 12th seconds from the time of consideration it is found to move through 12 and 10 feet respectively. What was the velocity of the particle at the beginning of the period of consideration and with what acceleration is it moving?

Ans. 21.5 ft. per sec., $-$ 1 ft. per sec. per sec.

32. Two stations A and B are M metres apart. A train starts from rest at A and is accelerated and comes to rest again at B. If the retardation be twice the acceleration, prove that the maximum speed attained by the train must be at a distance of $\frac{2}{3}$ M metres from A.

33. A velocity is changed to one of the same magnitude in a direction at right angles to itself. Find the change in velocity and the acceleration supposing the change occurs in 12 seconds.

Ans. $v\sqrt{2}$ ft./sec., $\frac{v}{6\sqrt{2}}$ ft./sec^2.

34. A bullet leaves the barrel of a gun with a velocity of 1100 feet per second. If the acceleration produced inside the gun remains constant, find the time the bullet took to travel through the barrel, 2 feet 9 inches long.

Ans. .005 second.

35. A burglar's car took a start with an acceleration of 10 feet per second per second. A police vigilant party came in a car at the spot at a velocity of 60 feet per second after $2\frac{2}{3}$ seconds and continued to chase the previous one with uniform velocity. Find the time when the police van overtakes the burglar's car. *Ans.* after $1\frac{1}{3}$ seconds run from the spot.

36. If two cars run under the conditions of the cars in the previous problem, find how many times and when they cross each other.

Ans. 2 times, one after $1\frac{1}{3}$ sec.
one after $5\frac{1}{3}$ sec.

37. Find the space traversed by a dropped body in the 7th and 10th second. *Ans.* 209.3 ft.; 305.9 ft.

38. A stone is projected vertically upward with a velocity of 60 metres per second. How many metres will it pass over in the fourth second from starting? Locate its position at the end of the 6th second.

Ans. 25.665 metres; 183.42 metres high.

39. A ball is thrown vertically upward with a velocity of 96.6 feet per second. Find the height it can rise. *Ans.* 144.9 feet.

40. How long will it take for a dropped stone to reach the foot of a tower 150 feet high. *Ans.* 3.052 secs.

41. A stone is dropped from the top of a tower 110 feet high and at the same instant another is projected vertically upwards from the ground along the same course so that it can reach the top. Find where will the two stones meet? *Ans.* 82.5 ft. above the ground level.

42. Find the velocity of projection of a body thrown vertically upwards if it describes the same distance in the 1st second as that traversed by a freely dropped body in the first 3 seconds. *Ans.* 161 ft. per sec.

43. A stone is dropped into the water of a well and the sound of the splash is heard after a period of 3.5 seconds. Assuming the velocity of sound to be 1200 feet per second, find the depth of the well. *Ans.* 187.2 feet.

44. Mail bag is dropped from an aeroplane at an instant when it is descending vertically at a rate of 20 feet per second at the height of 400 feet. Find after what time the bag will reach the ground of the aerodrome.

Ans. 4.4 secs.

45. A rocket hits a cocoanut in a tree 15 metres high with a velocity of 200 metres per second. Find the velocity of the projection.

Ans. 200.76 metres per sec.

46. A body is dropped from the top of a tower and it is observed that it covers 12/25th. of the total height at the last second of its flight. Find the height of the tower in feet. *Ans.* 207.9 feet.

47. A man drops a stone from the top of a tower 150 feet high into a deep well at its foot. If the sound of the clash between the water and the stone reaches the ear of the man 6.5 seconds after it is dropped, and if the sound travels at 1120 feet per second, find the depth of the well.

Ans. 429 feet.

48. A rocket is thrown vertically from the roof of a building 40 feet high and its stick reaches the ground 4 seconds after the explosion is seen. If the velocity of the rocket is zero when it burst, find how high it went. What is its velocity when it passes the level of the roof? Neglect small deviation in the path of motion. *Ans.* 217.6 feet from the ground, 118.4 feet per second.

49. A bullet is shot vertically upwards and passes the altitude of 1500 feet on its upward and downward path respectively at an interval of 30 seconds. Find the muzzle velocity and the height to which it reached. Neglect the length of the gun. *Ans.* 574.3 ft./sec., 5123.5 feet.

50. A body is thrown vertically upwards with a velocity of 60 ft. per second. If the acceleration due to air resistance is given by the equation, $f = .003\ v$, find in what time the body will reach the highest position.

Ans. 1.84 seconds.

51. Calculate with what velocity a stone may be projected at an angle of 45° to the horizon in order to have a range of 110 yards.

Ans. 103 ft. per second.

52. If a stone be projected with a velocity of 75 ft. per sec. in a direction $\tan^{-1}.75$ with the horizon, find the maximum height the stone attains. *Ans.* 31.45 ft.

53. A projectile is fired horizontally from a height 145 ft. from the ground level and reaches the ground at a horizontal distance of 1000 yards. Find the velocity of throw. *Ans.* 1000 ft. per sec.

54. Find the horizontal distance where a projectile will strike a ship in a sea if it be fired from a gun on the top of a cliff 500 feet high with a velocity of 1000 feet per second at an angle of 30° up the horizontal direction.

Ans. $5\frac{1}{4}$ miles.

55. If two shells are fired from an anti-air-craft gun with velocities of 600 and 900 feet per second respectively at an angle of 75° with the horizontal and if the second one overtakes the first when it reaches the maximum height, determine the time intervals between the two firings.

Ans. 6 secs.

56. Two gunners, side by side, in a battle-ship aims at an enemy cruiser 2 miles off. If the shells are delivered simultaneously and if the velocities are 800 feet and 1000 feet per second respectively and if both of them hit the target, find angles of projection of the two velocities.

Ans. 16.05° & 9.95° respectively.

57. A snipe is spotted by a boy at a distance of 100 feet from him and on the same level. The boy throws a stone to hit it hard with a velocity of 65 feet per second. Find, at what angle the stone should be projected with this velocity. *Ans.* 25°.

58. If the snipe of the previous problem would rest on a compound wall 12 feet high, find the angle of projection of the throw to hit it

Ans 34° & 63°.

59. With what initial velocities two balls may be projected from the same point at angles of elevation $\sin^{-1}\frac{1}{2}$ and $\sin^{-1}\frac{\sqrt{3}}{2}$ respectively to have the same horizontal range?

Ans same velocity

60. In the previous example if the angle of elevation be 20° and 30° respectively, compare the initial velocities

Ans 2nd one is 1 161 times the former one

61. If a cartridge which is fired from a gun at an elevation of $\tan^{-1} 1$ with a velocity of 1000 feet per second strikes a pigeon, rising at the instant of shooting vertically up with uniform speed from the ground floor of a house 500 yards away from the gun, find the velocity of the bird

Ans 368 ft/sec

62. From the slope of a hill inclined 30° with the horizon two balls are thrown up and down the slope respectively with an angle of projection of 45° up the horizon Compare their ranges

63. A boy throws a stone with a velocity of 100 ft per sec from the roof of a house 30 ft high aiming at an object beyond the compound, which is horizontally 100 feet away from the place The stone just passes over the compound wall 10 feet high Find the angle of projection and the distance of the wall

Ans 7 55° down the horizontal, 77 45 feet

64. From a motor lorry running at a speed of 30 miles per hour a cooly throws a stone vertically upward with a speed of 10 ft per sec Find after what time the ball will come to his hand again

Ans 625 secs

65 A bomber plane flies at an altitude of 1610 feet at the rate of 90 miles per hour A bomb is dropped to hit a target Find at what horizontal distance the target is resting from the plane

Ans 1320 feet

66. A train is running with a velocity of 30 miles per hour A passenger, sitting on his seat, takes a glass of water and extends it outside the window to wash his hand, but the glass slips down from his hand which is 8 05 feet from the ground Deduce an equation for the absolute velocity of the falling body neglecting the air resistance How far does the train travel by that time? If the air resistance produces a retardation of 30 feet per second per second on the falling body in a direction opposite to that in which the train runs, compare the relative positions of the man and the glass when it touches the ground.

Ans $y = 00831\ x^2$,
$s = 31\ 12$ feet,
Glass touches the ground
7 5 feet behind the man

67. A ball starting from rest slides down a 10 feet long 45° inclined terrace of a house. If the lower end of the terrace is 40 feet above the ground, find where the ball will first touch the ground.

Ans. At a horizontal distance of 17.72 feet from the lower end of the terrace.

68. A body slides down a smooth inclined plane making an angle of 30° with the horizontal. At an instant its velocity is 10 feet per second. How long will it take to slide 30 feet down from that instant? *Ans.* 1.41 sec.

69. After running on a level track with a constant velocity of 60 miles per hour a motor car approaches a smooth inclined plane 1 in 2. Find how far and how long the car will run before it comes to rest if the engine remains inactive. *Ans.* 240.5 ft.; 5.465 sec.

70. With what velocity must a body start to rise up an inclined plane 5 metres long in $1\frac{3}{4}$ seconds? The inclination is 1 in 4.

Ans. 500.3 c.m/sec.

71. A plane 289.8 feet long inclined to the horizon at an angle $\sin^{-1}\frac{1}{4.5}$ is divided into two parts in such a way that a body starting from rest at the top travels the first portion in half the time required to travel the second. Find out the times and distances. *Ans.* 3 and 6 secs.; 32.2 and 257.6 ft.

72. A particle is allowed to slide freely along the concave side of a vertical circle of radius 100 feet from a point where the radius makes an angle of 30 degrees with the vertical diameter. Find the velocity at the lowest point. *Ans.* 29.3 ft. per secs.

73. In a vertical circle chords end at the lowest point. Show that the velocity of a particle at the lowest point of a chord starting from rest at the top is $\sqrt{\frac{g}{r}}\,.\,l$ where r is the radius of the circle and l is the length of the chord.

74. Compare the angular velocities of the hour, minute and second hands of a clock.

75. If the linear velocity of a train, running along a circular track of 1000 feet radius, be 10 miles per hour, find the acceleration created towards the centre. *Ans.* .2151 ft./sec.^{2}

76. If a boy rotates a ball with a string 2 feet long in a horizontal plane and if the acceleration along the string towards the hand be 2 feet per second per second, find the angular velocity of the ball. *Ans.* 1 rad./sec.

77. A railway line changes its direction by 30° in an arc length of 785 feet. If a locomotive runs on the line with a constant speed of 30 miles per hour, what is the acceleration on it when it moves on the curved track?

Ans. 1.291 ft./sec.^{2}

78. A fly-wheel 4 feet in diameter is speeding up at an angular acceleration of 3 radians per second per second. When the peripheral speed of the wheel is 4 feet per second, find the absolute acceleration of the topmost point of the wheel.

Ans. 10 ft./sec.^{2}, $\theta = \tan^{-1} \frac{3}{4}$ with the radial acc.

79. A motor car is running with an acceleration of 2 feet per second per second. When its speed is 3 feet per second, find the absolute acceleration of (1) the highest point of the wheel tyre, and (2) when the point makes an angle of 30° upwards at the axis with the horizontal radial line in the direction of the motion of the car. The diameter of the extreme surface of the tyre is 3 feet.

Ans. 7.071′/sec.^{2}, $\theta = \tan^{-1} \frac{2}{3}$; 6.22′/sec.^{2}, $\theta = 84.5°$ with the direction of motion.

80. The peripheral speed of a pulley is 100 inches per second and the speed of a point on an arm at a radial distance of 30 inches from the rim surface is 40 inches per second. Determine the diameter and the angular speed of the pulley.

Ans. $d = 100$ in., $\omega = 2$ rad./sec.

81. If the pulley of the previous problem is accelerated from rest to a speed of 120 r.p.m. in 10 minutes, find the number of revolutions it makes in the second 5 minutes from the starting. Take the acceleration as uniform.

Ans. 300 revolutions.

82. In a circus party a cyclist develops a speed of $\frac{22}{3}$ feet per second at the end of the first turn of a circular track of 25 feet radius. At the end of the second turn it is found that the velocity changes to 11 feet per second with a constant acceleration. Determine the radial and tangential accelerations at the end of the second turn.

Ans. 4.84 ft./sec.^{2}, .214 ft./sec.^{2}

83. A train changes its velocity uniformly from 20 feet per second to 40 feet per second while running on a curved path of radius 1000 feet and describes a length of 1000 feet, Find the magnitude of the absolute acceleration of the train when its velocity is 40 feet per second.

Ans. .601 ft./sec.^{2}

84. A body is moving in a simple harmonic motion. At a distance of one foot from the mid-position the magnitude of its velocity is 10 feet per second and at a distance of 2 feet it is 5 feet per second. Find the periodic time. What will be the magnitude of the acceleration at a distance of 1.5 feet from the mid-position? What is the amplitude?

Ans. t = 1.25 secs.
$f = 37.5'$ per sec. per sec.
$a = 2.236$ feet.

85. If a particle moves in a simple harmonic motion of amplitude 15 centimetres and if the periodic time be 1.5 seconds, find the magnitudes of its velocities and accelerations, .1 second, .3 second and .5 second after it begins its motion from one extreme end. *Ans.* $v = 25.55$ c. m.

$f = 240.2$ c. m.

Determine other values.

86. The motion of a particle is simple harmonic of amplitude 6 feet. If the time for a complete oscillation be 5 seconds, find the time taken by the body in passing two points 5 feet and 2 feet from the centre of oscillation and on the same side of it. *Ans.* .36 second.

87. The motion of a reciprocating part of a steam engine which is assumed to move obeying approximately the law of simple harmonic motion, can be represented by the equation, $f = -16x$ with an amplitude of 9 inches. Determine the period of frequency and the frequency of the moving part. Also find the displacement, the velocity and the acceleration of the part after .1 second counting from the end of the amplitude.

Ans. 1.57 sec.,
.637 oscillations/sec.
$x = .82$ in.
$v = 14$ in./sec.
$f = 132.48$ in./sec.^{2}

PART II — KINETICS (Dynamics)

CHAPTER V

FORCE AND MOTION

Cause and Effect

67. Force. The term 'force' was not in use before Galileo (1564-1642). The weight of a body would mean at that time the pressure it produced on which it rested. Galileo first introduced the name 'force' for the weight of a body. He found that there was an acceleration in a falling body and it was the same for all the falling bodies. He attributed the cause of this acceleration to the force. That is, force may be said to be the cause of the change in the state of motion in a body. He, therefore, concluded from the phenomenon that the weight of a body or the force acting on a falling body was proportional to the mass and it could be measured by the mass and acceleration.

Isaac Newton (1642-1727) generalised this idea and declared that for all moving bodies having acceleration not only for the falling bodies the force could be measured with the acceleration.

In the *Principia* Newton defined force as follows: *The impressed force is the action exercised on a body so as to change its state of rest or of uniform motion in a straight line.*

This definition of force states that force is nothing but the *action* exercised to change the state of motion in a body but gives no idea for measuring it.

68. The definition is followed by three laws of motion :

First Law. *Every body continues in its state of rest or of uniform motion in a straight line unless it be compelled by some impressed force to change that state.*

Second Law. *The rate of change of momentum in a body is proportional to the impressed force and takes place in the direction in which the force acts.*

Third Law. *To every action there is always an equal and opposite reaction, i.e., the mutual action and reaction between two bodies are always equal and opposite.*

69. *First Law.* The law does not only give the definition of force stated by Newton but furthermore describes about the state of motion in a body when there is no impressed force.

In the application of forces it is found that motion can be created in some bodies with greater difficulty than that can be created in some other bodies. Also it is found that in bringing to rest some bodies in motion require more effort than some other moving bodies. Therefore, the bodies must possess a property by virtue of which they offer a resistance to take up any change in the state of motion. This property is called *inertia.* Inertia depends on the mass of a body and it is actually the property of mass.

70. **Mass.** *The quantity of matter in a body is its mass.* No satisfactory definition can be given for matter. It is just like time and space a primary conception of mind. Matter appeals to our senses most characteristically through the effort required to a sudden change of motion in it. Generally it can be said that matter is that which possesses the property of inertia. This property of inertia varies directly with the mass or the quantity of matter. Hence, mass is sometimes said to be the measure of inertia. The mass of a body depends on the volume and density of the body and is directly proportional to both of them. It is measured by the product of the volume and density.

71. **Density.** *The density is the mass contained in unit volume of a body.* Density is generally stated as grammes per cubic centimetre in the metric system. The engineers and the commercial people, however, generally use the expressions, 'pounds per cubic foot', 'pounds per gallon' etc. Now,

if m = mass of a body

v = its volume

and ρ = its density Then, $m = v.\rho$

72. **Unit of Mass.** A lump of platinum marked *PS* 1844, 1 lb. kept in the Standard Deparmtent of the Board of Trade at

Westminster is the British unit of mass and is called Pound Avoirdupois.

Another lump of platinum made by Borda in 1795 is kept at Paris, which is the unit of mass according to the metric system, This is called a *Kilogramme.*

These units are called the absolute units. In the British system the unit is quite an arbitrary one ; but in the French system the lump was chosen by Borda perhaps to represent a mass equal to 1000 *c.c.* of distilled water at a temperature of 4° *C*. Actually a kilogramme of water at that temperature occupies 999.97 *c.c.* Still, however, the mass of 1000 *c.c.* of distilled water at 4° *C* is taken to be 1 kilogramme which is sufficiently correct for most of the purposes. It is taken as such for a great advantage in numerical calculations. For scientific purposes *a gramme which is a thousandth part of a kilogramme is taken as the unit.*

One pound is equivalent to .4536 kilogramme or 453.6 grammes.

73. *Second Law.* The first law states the action of force on a body, while the second law devises the method of measuring the force. The second law states that the rate of change of momentum is proportional to the force applied.

74. **Momentum.** *Momentum or the quantity of motion (in Newton's phrase) of a body is measured by the product of its mass and velocity.* If m be the mass of a body and v its velocity, then $m.v$ is the measure of the momentum of the body.

75. **Unit of Momentum.** Momentum directly varies as both the mass and the velocity. Therefore, it directly varies as the product of m and v, *i.e.*, Momentum $\propto m.v$. Hence, Momentum $= k.m.v.$, where k is a constant. If a quantity be called unit momentum when a body of unit mass possesses unit velocity, then the constant k becomes equal to unity, and momentum $= m.v$.

The absolute units in two different systems are pound-foot per second and gram-centimetre per second respectively. The gravitational units have no definite names and are represented by units alone, such as, one unit, two units, and so on. The gravitational units will be understood hereafter.

76. **Unit of Force.** From the second law of motion, Rate of change of momentum $\propto P$, where P is the impressed force at an

instant, *i.e.*, during a very small interval of time dt including that instant.

Hence, $P \propto m \times$ rate of change of velocity (mass remaining constant).

i.e., $P \propto m \times \frac{dv}{dt}$, where dv is the small change in velocity in small time dt.

$$i.e., \quad P = k.m \frac{dv}{dt}, \text{ where } k \text{ is a constant.}$$

Now, if the quantity of P be so chosen that it acting on unit mass creates unit $\frac{dv}{dt}$, then k becomes unity and this quantity of force is taken as the unit force, and the equation may be put as

$$P = m.\frac{dv}{dt} = m.\frac{d^2s}{dt^2} = m.f \qquad \cdots\cdots\cdots\cdots Eq.\ 30$$

The 'f' in the equation represents the acceleration at an instant when the force acting is P. In case where P is a constant force, f is the constant acceleration created on the mass due to the action of the accelerating force P.

Thus, *the unit of force is defined as the quantity of force that acting on unit mass creates unit acceleration.*

In British system (Foot-Pound-Second or F. P. S. system) the unit of force is known as a *poundal.* One poundal of force acting on a mass of one pound creates an acceleration of one foot per second per second.

In French system (Centimetre-Gramme-Second, or C.G.S. system) the unit of force is a *dyne*, which acting on a mass of one gramme creates an acceleration of one centimetre per second per second.

These units are known as absolute units.

77. Law of Gravitation. *Every particle of matter in this universe attracts every other particle with a force proportional to the product of their masses and inversely proportional to the square of the distance between them.*

As each particle is being attracted by all other particles in this universe, the acceleration produced on a particle as is observed, is the resultant of all the accelerations created by the attractions of those particles on it.

If there are two particles of masses m_1 and m_2 respectively at a distance of a, the mutual attraction is proportional to $\frac{m_1 . m_2}{a^2}$ and m_1 has an acceleration towards m_2 by an amount proportional to $\frac{m_2}{a^2}$ and m_2 has an acceleration towards m_1 by an amount proportional to $\frac{m_1}{a^2}$. Newton asserts that this is true for every pair of particles in this universe. He calculated the motions of the planets and satellites on the assumption of this law and the results of calculation agreed with the experimental observations.

78. Weight. Every particle on the surface of the earth is being attracted by the mass of the earth towards its centre. The earth is approximately a spherical body and its total mass is assumed to be concentrated at the centre without any appreciable error. The attraction on a particle by the earth towards its centre is known as the attraction due to gravity or the *weight* of the particle. The acceleration produced due to this attraction is not constant throughout the surface of the earth, because the mutual attraction which is proportional to $\frac{m_1 \; m_2}{a^2}$ depends on the value of a, the radius of the earth, which varies with different places. The polar diameter of the earth is less than its equatorial diameter, for which the acceleration varies from 978.1 c.m. or 32.09 feet per second per second at the equator to 983.11 c.m. or 32.354 feet per second per second at the pole.

Thus, the weight of a mass is the force with which the earth attracts it and it is directly proportional to the mass. For a given place the weight of a mass always remains constant, but with different places it slightly varies. *Though the mass does not change, the weight changes.*

79. Unit of Weight. The unit of weight in British system is the attraction of earth on a mass of one pound and is called pound's weight or a force of one pound. In C. G. S. system the unit is a kilogramme's weight, *i.e.*, the attraction of earth on a mass of one kilogramme.

80. A table showing the value of the acceleration at different places on the surface of the earth is given below :

Place		Value of 'g' Foot-Second Unit	Centimetre-Second Unit
Pole	...	32.254	983.11
Edinburgh	...	32.203	981.54
Berlin	...	32.194	981.25
Greenwich	...	32.191	981.17
Paris	...	32.183	980.94
Calcutta	...	32.113	978.81 *
Equator	...	32.090	978.10

81. Engineers are to deal with big quantities of force, for which the absolute units, poundal and dyne, are very small for measuring them. They chose bigger units for the purpose. These units are known as the gravitational units or the engineers' units. These units are the weights of unit masses in the two systems respectively.

82. Gravitational Units of Force and Mass. It is found that the weight of a body creates an acceleration g. Therefore, if the weight of unit mass is taken as unit of force, then, this unit acting on unit mass will create an acceleration .. .. $= g$

or .. twice the unit mass .. .. $= \frac{g}{2}$

or .. thrice .. .. $= \frac{g}{3}$

and so on.

Thus, if the mass be taken as g times the absolute unit, it is found that this new unit of force acting on that mass creates a unit acceleration. The engineers change, therefore, to their advantage, the unit of force as well as the unit of mass accordingly without altering, in any way, the definition of unit of force. These new units are called the *gravitational* units or the *engineers'* units. The gravitational unit of mass has no definite name and it is represented by units alone.† Thus, both the gravitational units of force and mass are 'g' times the units in absolute system respectively.

* The figure is obtained from the Survey of India office (Geodetic Branch) —Dehra Dun.

† The terms, 'slug' and 'gee-pound' are being recently used in place of 'gravitational unit of mass', *e.g.*, one gravitational unit of mass, two gravitational units of mass, etc. are named as 1 slug or 1 gee-pound, 2 slugs or 2 gee-pounds, and so on.

In London, where the value of g is 32.2 ft. per sec. per sec. or 981 c.m. per sec. per sec. approximately, a pound's weight is, therefore, equal to 32.2 poundals, and the weight of a gramme is equal to 981 dynes. The mechanical engineers' unit of force in British system is a pound's weight and in Metric system is the weight of a kilogramme which is 1000 times a gramme's weight and is equal to 9.81×10^5 dynes. The electrical engineers' unit is always a dyne. For scientific purposes dyne is the unit of force.

It is to be noticed that the absolute units of force, poundal and dyne, were chosen depending on the mass and the acceleration produced in it. But the gravitational units were chosen arbitrarily without paying any attention to the effect it produced on the mass. Considering the effect of these units, the units of mass are altered obeying the definition of the unit of force.

83. Thus, according to these new units of force and mass, the form of the equation becomes $P = \frac{m}{g} f$, where m is the mass in absolute unit. This can also be put as $P = \frac{w}{g} f$, without any numerical error, where w is the weight of the mass, because the weight of a mass is numerically equal to the magnitude of the mass in absolute system, *i.e.*, $w = m$. When we say that the weight of a body is x lbs., we actually mean the quantity of force that is acting on a mass of x lbs. Thus, both the weight and the mass are represented by x units.

The symbolic representation of the relation amongst m, v and ρ is $m = v.\rho$ and can, therefore, be put in the form, $w = v\rho$, *i.e.*, the word 'weight' can be substituted for the word 'mass' in the definition of density without any numerical error. Generally a body is represented by its weight and not by its mass. Therefore, the density of a body is defined by some writers, specially the engineers, as the *weight per unit volume.*

ACTION OF SIMULTANEOUS FORCES ON A PARTICLE

84. Physical Independence of Force. Suppose several forces are acting simultaneously on a particle either at rest or in motion. The second law of motion being true for each and every force, the effect of each of these forces is completely determined by the law. Each of these forces must have its own effect on the particle in so

far as the change of motion is concerned, as if, it were the only force acting on the particle. Thus, each of these forces will create an acceleration following the relation $P = m.f$ in its own direction. That is, the effect of each force on the particle is independent of other forces. This is known as the *Physical Independence of Force.*

85. Force is a vector quantity, because it has both magnitude and direction and can be fully represented by a vector. The resultant of all the forces acting simultaneously on a particle can, therefore, be found out by compounding all the forces vectorially and the acceleration due to this resultant force can easily be obtained, or the acceleration due to each of the forces is found out separately and then by compounding vectorially all these accelerations the resultant acceleration is obtained. Whether the accelerations will be found out individually first and then be compounded, or the forces will be compounded first and then the acceleration due to the resultant force is to be solved is immaterial.

86. Compounding of Forces. For the present only the forces acting on a particle in the same plane, *i.e.*, the concurrent and co-planer forces will be dealt with. The methods of compounding are just similar to those by which the velocities were compounded. A force may be said to possess four elements : (1) Magnitude, (2) Direction, (3) Sense and (4) Point of Application. Most of the authors combine direction and sense into one element and denote it by the term 'direction' only. It is to be marked that in case of a particle the forces are concurrent because the point of application is a definite single point. The methods are :

1. *Vector addition.*

2. *Triangle of forces.* If two simultaneous forces acting on a particle can be represented in magnitude and direction by the two sides of a triangle, the third side represents the *resultant*, its direction being always towards the last side drawn in succession.

3. *Parallelogram of forces.* If two simultaneous forces acting on a particle can be represented in magnitude and direction by the two adjacent sides of a parallelogram, the resultant of the forces is represented in magnitude by the diagonal of the parallelogram passing through their point of intersection, the direction being away from the point of intersection.

4. *Method of resolution.* Each of the forces is resolved in two definite directions at right angles to each other, generally, along two axes—vertical and horizontal. The resultant, $R = \sqrt{\Sigma H^2 + \Sigma V^2}$, where ΣH and ΣV are the sums of the horizontal and vertical components respectively. If θ be the inclination of the direction of the line of action of the resultant force with the horizontal direction, $\tan = \theta \dfrac{\Sigma V}{\Sigma H}$.

The third method can be directly proved in case of velocities, but in case of forces this cannot be directly proved. It only agrees with the results of different experiments. Hence, the proof is a deductive one. Parallelogram of accelerations helps to explain the case of parallelogram of forces.

87. Impulse. $P = m. \dfrac{dv}{dt}$ (from *Eq.* 30)

or, $P. dt = m. dv$

By integration within the limits (in time t, velocity changes from v_1 to v_2) assuming P to be constant,

$$\int_0^t P. dt = \int_{v_1}^{v_2} m. dv$$

or, $P t = m (v_2 - v_1)$ .... *Eq.* 31

or, it may be treated as follows :

$$P = mf = m \left(\frac{v_2 - v_1}{t} \right)$$

or, $P \times t = m (v_2 - v_1)$

The symbolic representation means that if a constant force P acts on a mass m for t seconds, the velocity of the mass changes from v_1 to v_2. The product of P and t is called the *impulse* of the force P and is equal to the change of momentum in a particle. Thus, the second law of motion can also be defined as follows :

The change of momentum is equal to the impulse of the impressed force and is in the same direction with it.

If a constant force P acts on a particle whose mass is m and if the particle is initially at rest, the impulse $Pt = m. v$, where v is the velocity attained by the particle in time t.

88. Momentum from Force-Time Curve. As the space described by a particle was found out from the velocity-time curve (Art. 15), the impulse of a force, too, can be determined in the same way from the force-time curve. The total area under the curve represents the amount of the impulse or the total change of momentum.

89. Time Average Force. In case of varying force if the total impulse or the change of momentum be divided by the time for which the force is impressed, the quantity obtained is the *time-average force.*

90. Impulsive Force. When a very big force acts on a mass for a very short period of time to produce a great change in momentum, the force is said to be an *impulsive force.* Instances are found in cases of collisions, blows and impacts.

Illus. Ex. 37. *If a constant accelerating force of 10000 dynes acts on a body weighing 30 kilograms for 10 seconds, find the velocity of the body at the beginning of the 11th second.*

If P_f be the accelerating force, $P_f \times t = m \times v$,

$P_f = 10000$ dynes

$t = 10$ seconds

$m = 30 \times 1000$ gms

v in 10 sec $= ?$

therefore, substituting the values,

$10000 \times 10 = 30 \times 1000 \times v$

$$v = \frac{10000 \times 10}{30 \times 1000} = 3.33 \text{ c.m. per sec}$$

Illus. Ex. 38. *The engine of a motor car weighing 500 kilograms exerts a force of 35 kilograms weight to speed it up against a constant resistance of 40 grammes per kilogram weight of the car. Find the time in which it attains a velocity of 7 metres per second from rest.*

Let P_T represents the total force and R the resistance, here,

$P_T = 35 \times 1000$ gms wt

$R = 500 \times 40$ gms wt

$\therefore$ P_f (accelerating force) $= P_T - R = 35000 - 20000 = 15000$ gms. wt.

$v = 7 \times 100$ c.m. per sec.

$t = ?$

Substituting the values of P_f, m & v in $P_f t = m.v$

$$15000 \times t = \frac{500 \times 1000}{981} \times 7 \times 100,$$

from which $t = 23.75$ secs.

Illus. Ex. 39. *A cyclist is applying a force of 200 poundals on the pedal. If the total weight of the cycle with the man be 14 stones, find the acceleration produced. In what time will the velocity be 15 miles per hour? Neglect the frictional and other resistances.*

(I)

$P_f = 200$ poundals.

$m = 14 \times 14 = 196$ lbs.

$f = ?$

$P_f = m.f$ Substituting the values of P_f and f,

$200 = 196 \times f.$

$\therefore\ f = 1.02$ feet per sec. per sec.

(II)

$u = O$

$v = 22$ ft. per sec.

$f = 1.02$ ft. per sec. per sec.

$t = ?$

Substituting the values of the available data in, $v = u + f.t$

$22 = 1.02 \times t$, or $t = 21.5$ secs.

OR

$P_f = 200$ poundals

$m = 196$ lbs.

$v = 22$ ft. per sec.

$t = ?$

From $P_f \times t = m \times v$

$200 \times t = 196 \times 22$

$\therefore\ t = 21.5$ seconds.

Illus. Ex. 40. *A train weighing 200 tons rises up a slope of 1 in 150. The draw-bar pull of the engine is 10 tons and the frictional and other resistances amount to 20 lbs. per ton weight of the train. Starting from rest the train goes up 2 miles coming to rest again at the end of the travel without the use of brakes. Find out the time required.*

Total force, $P_T = 10 \times 2240$ lbs. wt.

Resistance due to friction etc., $R_1 = 20 \times 200 = 4000$ lbs. wt.

Resistance due to the inclination of the plane,

$$R_2 = \frac{200 \times 2240}{150} = 2987 \text{ lbs. wt.}$$

Therefore, $R\,(\text{Total}) = (R_1 + R_2) = 6987$ lbs. wt.

and $P_f = P_T - R\,(\text{Total}) = 22400 - 6987 = 15413$ lbs.

Space traversed, $s = 2 \times 5280$ ft.

$$\text{Mass, } m = \frac{200 \times 2240}{32.2} \text{ units.}$$

Now, during the time of travel there must be a period of acceleration as well as a period of retardation.

The average velocity during the run $= \frac{s}{t} = \frac{5280 \times 2}{t_1 + t_2}$, where t_1 and t_2 are the periods of acceleration and retardation respectively and $t = t_1 + t_2$

Therefore, the maximum velocity during the run $= 2 \times \frac{5280 \times 2}{t_1 + t_2}$

Again, the change in momentum, though opposite in direction,

in both the periods $= \frac{200 \times 2240}{32 \cdot 2} \times 2 \quad \frac{5280 \times 2}{t_1 + t_2}$

But $P_f \times t_1 = \frac{w}{g} v = P_{-f} \times t$, where P_{-f} is the retardation

$$\text{or,} \quad 15413\, t_1 = 6987\, t_2 = \frac{200 \times 2240 \times 2 \times 5280 \times 2}{32 \cdot 2\,(t_1 + t_2)}$$

From the first two expressions, $t_1 = \frac{6987}{15413} t_2$

$$\therefore \quad t_1 + t_2 = t_2 \left(1 + \frac{6987}{15413}\right) = t_2 \times \frac{22400}{15413}$$

$$\text{or } t_2 = \frac{15413}{22400} (t_1 + t_2)$$

Now, substituting the value of t_2 in the second expression in terms of $t_1 + t_2$,

$$6987 \times \frac{15413}{22400} (t_1 + t_2) = \frac{200 \times 2240 \times 2 \times 5280 \times 2}{32 \cdot 2\,(t_1 + t_2)}$$

$$\text{or,} \quad (t_1 + t_2)^2 = \frac{200 \times 2240 \times 2 \times 5280 \times 2 \times 22400}{32 \cdot 2 \times 6987 \times 15413} = 61100$$

$$\therefore \quad (t_1 + t_2) \text{ or } t = 247 \text{ seconds} = 4 \text{ min } 7 \text{ secs}$$

91. *Third Law*. This law is deduced from experiments and observations. In the enunciation, the word 'action' means the exertion of force, *i.e.*, the creation of momentum The third law states that if a particle exerts a force on another particle, then the second particle will also exert an equal amount of force on the first just in the opposite direction This mutual action and reaction can easily be felt when a push is given on a wall. One will be pushed back as soon as he gives a thrust on a wall.

This is to mention here that the third law of motion is also true for the cases of electrical and magnetic attraction and repulsion. In fact, this law is true for all mechanical actions.

From the third law it is clear that each and every force in action, we deal with, is one of a pair of equal and opposite forces.

A simple experiment to show that actions and reactions are equal may be performed with a cord fixed at one end with a wall-hook and running horizontally over two frictionless pulleys placed at a distance. A weight is suspended at the free end. The system is in equilibrium. If, now, the fixed end of the cord is made free from the wall, it is found that the amount of weight that should be suspended at that end to keep the system in equilibrium is equal to the weight suspended at the other end. Therefore, it is concluded that the wall impresses the same amount of force through the medium of cord as the suspended weight exerts on the wall.

If a body be placed on a table, the body will impress its own weight on the table, whereas the table also will exert the same amount of force on the body just in the opposite direction. The state of rest explains the condition.

If a body in motion comes in contact with another body in motion or at rest, then the first body gives a thrust on the second body for which the second body gains a quantity of momentum which depends on the duration of the application of force. But by the law the same amount of force will be acted on the first body by the second one for the same period just in the opposite direction creating an equal amount of momentum in that direction. Thus, the momentum of the first body is reduced by the amount the second body gains. The total quantity of momentum in the system remains constant.

Take the case of a gun. When it is fired the gun kicks the man behind. Due to the explosion of the gas, force acts on the bullet to push it out of the barrel, and also according to the third law, an equal amount of force acts on the gun just in the opposite direction, and hence the backward push is felt. The momentum of the bullet is theoretically equal to the momentum of the gun (same amount of force is acting on both the bullet and the gun for the same period). It can be symbolically represented as, $MV = mv$, where M and m are the masses and V and v are the velocities of the gun and the bullet respectively. The velocities are, therefore, inversely proportional to their masses. The mass of the gun being much greater than the mass of the bullet, the velocity of the recoil is much less than the velocity of the projectile. The total quantity of momentum in the system remains unchanged. The momenta being opposite in direction, if

one of them be taken as a gain, the other one must be taken as a loss.

92. Whenever there is a change of momentum in two masses in a system due to the mutual action and reaction between them, the loss in one is equal to the gain in the other, *i.e.*, the total quantity of momentum in the two masses remains unaltered by any mutual action and reaction between them. This principle is known as the *Conservation of Momentum.*

93. **Definition of Momentum.** If a mass comes in contact with another mass having momentum, it gains motion whereas the other one loses. This propagation of motion from the latter to the former mass is due only to the existence of momentum in the second mass. No other cause can be ascertained for the phenomenon. Momentum may, therefore, be defined as follows:

Momentum is the property of mass at a definite state (when it has got motion) by virtue of which motion can be transferred from one body to another, when they are brought in contact with each other.

As this property is directly proportional to both the mass and velocity, it is measured by the product of the mass and velocity. It is a vector quantity.

94. When two bodies are unimpeded in their action on each other, that is to say, if the two bodies can be isolated from the influence of other forces, then it is found that not only the gain of momentum in a body is equal to the loss of momentum in the other, but the rate of gain in one is equal to the rate of loss in the other. Suppose, a body whose momentum is $m_1 v_1$ exerts force on another body having a momentum $m' v'$. If the amount of force be P and if it acts for time t, to change the velocities of the masses from v_1 to v_2 and from v' to v'' then,

$$Pt = m_1 (v_2 - v_1) = m' (v'' - v').$$

$$\text{or, } P = \frac{m_1 (v_2 - v_1)}{t} = \frac{m'(v'' - v')}{t}$$

$$\text{or, } P = m_1 f_1 = m' f',$$

where f_1 and f' are the rates of change in the two masses respectively.

From astronomical observations it is found that the mutual attraction between the earth and the moon has action in such a manner that $m'f' = mf$, where m' and m are the masses of the earth and the moon and f' and f are the accelerations created on them respectively. The accelerations are, therefore, inversely proportional to their masses.

95. Action and Reaction between two bodies under different conditions:

1. Two masses suspended about a frictionless pulley:— Suppose two masses of weights W_1 and W_2 are suspended about a frictionless pulley by means of a light inextensible thread as shown in the diagram (Fig. 49). Let W_1 be greater than W_2. The pulley will then move clockwise as shown by the arrowhead. Let T be the tension in the string, which is nothing but the mutual action and reaction between the two bodies. First, consider the condition of each of the masses separately. In case of the mass weighing W_1, the force W_1 (gravitational unit) is acting downwards and the tension T is acting upwards, but as the mass is going downwards, W_1 must be greater than T, and therefore, $W_1 - T = \frac{W_1}{g} f$ (i), where f is the acceleration in the system. Similarly, in case of the mass weighing W_2, T is greater than the force W_2, because the mass is going upwards, and therefore, $T - W_2 = \frac{W_2}{g} f$ (ii)

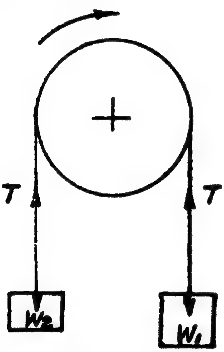


FIG. 49

Now, adding the two equations (i) & (ii), $W_1 - W_2 = \frac{W_1 + W_2}{g} f$

$$\text{or } f = \frac{W_1 - W_2}{W_1 + W_2} g \qquad \cdots\cdots\cdots Eq.\ 32$$

Substituting the value of f in (i),

$$T = \frac{2 W_1 W_2}{W_1 + W_2} \qquad \cdots\cdots\cdots Eq.\ 33$$

Again, if the whole system is considered all at once, the acceleration can be found out in the following way : The weight W_1 being greater than the weight W_2, and being opposite in direction, the accelerating force in the system is $W_1 - W_2$, which creates motion in the system in a direction as shown by the arrow-head in Fig. 49.

$$\text{Thus, } W_1 - W_2 = \frac{W_1 + W_2}{g} f,$$

$$\text{or, } f = \frac{W_1 - W_2}{W_1 + W_2} g$$

It is to be noticed here that the mutual action and reaction, T, is neither equal to W_1 nor to W_2, but is equal to a value which is greater than W_2 but less than W_1. The function of the force W_2 in this case is found to be against the creation of motion in the system as if it is acting like a resistance against the motion. This kind of resistance is found in many cases in practical fields to appear as friction, which is treated elaborately in the subject of Statics. If the case of a railway engine and the composition of carriages is considered, the engine exerts a force on the composition of bogies to drag it but the composition offers a resistance due to friction between the axles and the bearings. Thus, if P be the force applied by the engine and R be the resistance offered by the bogies, the accelerating force is $(P - R)$. Therefore, the acceleration is $\frac{P - R}{w + W} g$, where w and W are the weights of the engine and the bogies respectively.

2. The mass whose weight is W_2, instead of being suspended as in the previous case, rests on a perfectly smooth horizontal table top

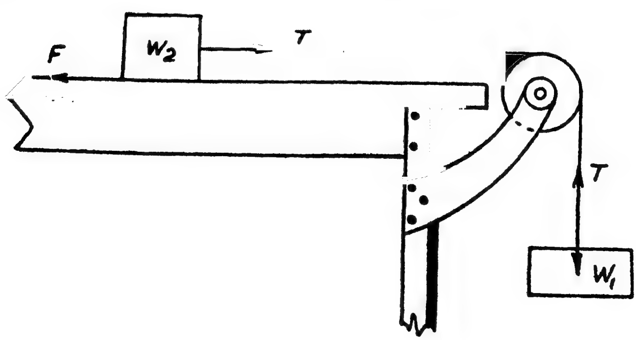


FIG. 50

(Fig. 50). Here also, treating in the same way as was done in the previous case, the accelerating force on the mass weighing W_1 is $W_1 - T$, and therefore,

$$W_1 - T = \frac{W_1}{g} f \qquad \text{(i)}$$

Again, the accelerating force on the mass weighing W_2 is only T, as the surface of the table has been said to be perfectly smooth.

Therefore, in this case, $T = \dfrac{W_2}{g} f$ (ii)

Adding (i) & (ii), $f = \dfrac{W_1}{W_1 + W_2} g$ .... *Eq.* 34

Substituting this value of f in the equation (ii)

$$T = \frac{W_1 W_2}{W_1 + W_2} \qquad \text{........} \textit{Eq. } 35$$

which is just half the value obtained in the previous case.

But if there is a resistive force, F, acting against the motion then

$$f = \frac{W_1 - F}{W_1 + W_2} g, \qquad \text{........} \textit{Eq. } 36$$

$$\text{and } T = \frac{W_1 (W_2 + F)}{W_1 + W_2} \qquad \text{.......} \textit{Eq. } 37$$

In case like this the resistance generally appears from friction. Friction is a thing to be discussed elaborately in Statics. For the present it is sufficient to know that 'friction' is the property of two bodies in contact by virtue of which a resistance is offered to any sliding motion between them.

Illus. Ex. 41. *A hammer weighing W lbs. after falling, due to gravity alone, from a height of h feet strikes a pile weighing W' lbs. and does not rebound. If the pile is driven x feet into the ground, show that the average force due to the blow is,* $\dfrac{W^2}{W + W'} \cdot \dfrac{h}{x}$ *lbs., and determine the average resistance of the ground.*

In falling from a height of h feet, the velocity of the body $= \sqrt{2gh}$.

Momentum before the impact is equal to the momentum after the impact.

Therefore, $\frac{W}{g}\sqrt{2gh} = \frac{W + W'}{g} V'$, where V' is the velocity of the combined body after the impact.

$$V' = \frac{W}{W + W'} \cdot \sqrt{2gh}$$

If the depth of penetration be x feet, then the velocity at the end being zero,

$$f = \frac{W^2}{(W + W')^2} \cdot \frac{2gh}{2x}$$

Now, if P be the average force of the blow, which is equal to the average resistance to penetration,

$$P = mf = \frac{W + W'}{g} \cdot \frac{W^2}{(W + W')^2} \cdot \frac{2gh}{2x}$$

$$= \frac{W^2}{W + W'} \cdot \frac{h}{x} \text{ lbs wt.}$$

Again, $v^2 = 2gh$, in case of falling bodies (in this case the hammer). Therefore,

$$h = \frac{v^2}{2g}$$

If R be the average resistance of the ground,

$$R = P + (W + W')$$

$$= \frac{W^2}{W + W'} \cdot \frac{h}{x} + (W + W')$$

$$= \frac{W^2}{W + W'} \cdot \frac{v^2}{2gx} + (W + W')$$

Illus. Ex. 42. *A hammer weighing 5 lbs. falls freely from a height of 4 feet and strikes a nail weighing 2 ounces driving it through 1.25 inches into a block of wood. What is the impulse of the blow? Find also the average force of the blow and the duration of penetration.*

The velocity of the hammer after falling from a height of 4 feet $= \sqrt{2 \times 32.2 \times 4}$

$\therefore$ the impulse $= \frac{5}{32.2} \sqrt{2 \times 32.2 \times 4} = 2.5$ units.

Again, P (according to the previous example) $= \dfrac{25}{5\frac{1}{8}} \times \dfrac{4}{\frac{5}{4 \times 12}}$

$= 187.5$ lbs.

But, $Pt = 2\frac{1}{2}$, $\therefore t = \frac{5}{2 \times 187.5} = .01$ second.

Illus. Ex. 43. *A hammer weighing 2 lbs. strikes a nail weighing 1 oz. to penetrate it through a partition wall with a velocity of 16 ft. per second and does not rebound. Find the average resistance offered by the wall if the penetration be 0.25 inch.*

Because the nail and the hammer move horizontally, their weights do not press the wall, *i.e.*, their weights must not be added with the value of the average force of the blow.

Taking the nomenclature of Illus Ex 41 and substituting the numerical values,

$$R = \frac{4}{2\,\frac{1}{16}} \cdot \frac{\cancel{16} \times 16}{\cancel{2} \times \cancel{32.2} \times 25}$$

$$= 31.26 \text{ lbs.}$$

Illus. Ex. 44. *What is the acceleration in a body weighing 7 lbs. and moving up a perfectly smooth plane 1 in 2 with the horizon due to the suspension of a weight of 5 lbs. about a frictionless pulley fixed at the ridge of the plane? Find also the tension in the cord connecting the body with the weight.*

$$f = \frac{5 - 7 \times \frac{1}{2}}{5 + 7} \times 32.2 \quad (Eq.\ 36)$$

$$= 4.025 \text{ ft. per sec. per sec.}$$

$$\text{and } T = \frac{5\left(7 + \frac{7}{2}\right)}{5 + 7} = 4.375 \text{ lb. wt.} \quad (Eq.\ 37)$$

Illus. Ex. 45. *Compare the pressures felt on the palm of a hand if a mass of 1 lb. be placed on it,* (*I*) *while at rest* (*II*) *while raising the hand up with an acceleration of 2.2 ft per sec. per sec. and* (*III*) *when the palm is moved downwards with the same acceleration.*

While at rest the downward force must be equal to 1 lb. wt.

When it is going up, the upward force will be such that it can create an acceleration of (32.2 + 2.2) ft. per sec. per sec.

Therefore, the force is equal to,

$$\frac{1}{32.2} \times 34.4 = 1.07 \text{ lbs. wt.}$$

By the third law the upward force must be equal to the downward pressure, and therefore, it is equal to 1.07 lbs. wt.

While moving downwards the natural acceleration due to gravity is destroyed by an amount equal to (32.2 — 2.2), *i.e.*, 30 ft. per sec. per sec. by the application of a force in the upward direction. This force must be equal to,

$$\frac{1}{32.2} \times 30 = .933 \text{ lbs. wt.}$$

The upward force being always equal to the downward pressure, the pressure on the palm in this case $= .933$ lbs. wt.

N.B. *It is to be noticed then, that if the palm be moved with an acceleration* 'g' *downwards, no pressure will be felt on it.*

96. Atwoods Machine. It consists of a firmly fitted vertical stand about 250 c.m. long with a graduated scale attached by its side. At the top of the stand there is a platform, on which there is a fitting arrangement for a light grooved pulley to rotate most freely. Two equal weights are suspended over the grooved pulley just like the arrangement as is shown in diagram (Fig. 49), by means of a fine cord. Motion is created in the system by the addition of a small weight called a rider, with one of the two weights. A clock-work to measure time to tenths of a second is required.

With the help of this instrument laws of motion under gravity can be verified and hence the value of 'g' of a place can be found out. Also the formulae of the article 32 can be established. But various corrections are required for accurate results and, therefore, this method, which gives only an approximate value, is not adopted for measuring the actual value of the acceleration g.

The method of finding out the value of 'g' accurately will be described during the discussion of a simple pendulum.

FORCE PRODUCING UNIFORM CIRCULAR MOTION

97. Centripetal and Centrifugal Force. The second law of motion states that while there is an acceleration in a particle there must be a force acting to produce that acceleration. It was proved that the acceleration produced in uniform circular motion is

$$\omega^2 r, \ \frac{v^2}{r} \quad \text{or} \quad v\omega$$

towards the centre (Art. 54). The force that creates an acceleration by that amount in the specified direction is called the centripetal force. If w be the weight of the particle then the magnitude of the force

$$P = \frac{w}{g}\omega^2 r, \ \frac{w}{g}\cdot\frac{v^2}{r}$$

$$\text{or} \ \frac{w}{g} v\omega \text{ units (gravitational).} \quad \cdots\cdots \quad Eq.\ 38$$

Again, according to the third law of motion, when this centripetal force acts on a particle towards the centre through some agent, the particle also will exert an equal amount of force through the same *agent on the centre just in the opposite direction. This force in action which is equal in amount to the centripetal force but opposite in* direction is called the centrifugal force. Thus, when a particle rotates uniformly in a circle about a definite point two equal and opposite forces appear to act in the particle—the one, towards the centre, is called the *centripetal force* and the other, in a direction away from the centre, is called the *centrifugal force.*

98. The agent or the medium through which the above forces act may be of various kinds. The instance of a direct medium is found in the case of a ball being rotated by means of a string. The forces act through the medium of string keeping it tight. If a man holds one end of the string with which a ball is rotated, he can directly feel the centrifugal tension.

But there are cases where a direct medium is not available for the application of forces; some mechanical device is made for the purpose.

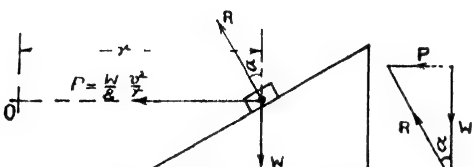


FIG. 51

99. Banking of a curved track. When a particle moves with a constant velocity on a horizontal plane, only two forces act on the particle—its own weight, W, and the normal reaction, R, (which are equal and opposite in direction). But if this particle is to rotate about a definite point, O (Fig. 51) a force must be brought into play to create an acceleration of $\frac{v^2}{r}$ towards O, where r is the radius of the circular path. There being only two forces W and R acting on the particle when on horizontal plane, these two forces

combined together cannot create any effect towards O, because they are equal and opposite, and the resultant must, therefore, be zero. Hence, if R be inclined, *i.e.*, if the plane on which the particle moves be banked, then this inclination of the plane can be made in such a way that W and R can produce a resultant towards O by the necessary amount to move the particle in the circular path. In the vector diagram (Fig. 51) $W + R = P$, the necessary amount of force, and it is equal to $\frac{w}{g} \cdot \frac{v^2}{r}$, which is the condition for a particle to rotate in a circle. Now in the vector triangle,

$$\tan \alpha = \frac{P}{W} = \frac{W}{g} \cdot \frac{v^2}{r} \div W$$

$$= \frac{v^2}{gr}, \text{ where } \alpha \text{ is the angle of inclination.} \qquad Eq.\ 39$$

100. Super-elevation of the railway track or roads at the point of turning. In the case of laying the railway lines and constructing motor tracks it is difficult to make use of the value of α directly. In those cases the difference of levels of the two ends of the track, breadth-wise, are maintained according to the value of α. Let the case

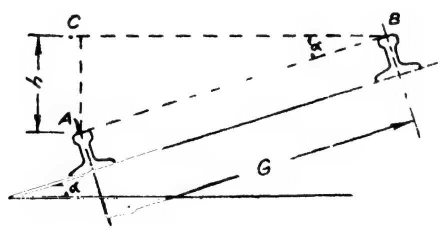


FIG. 52

of a circular railway track be considered. The diagram (Fig. 52) is the sectional end view of the track drawn in an exaggerated scale. The distance between the two lines, AB, which is called the gauge of the line, is equal to G (say). Let v be the velocity of the train and r be the radius of the curved path. AC, which represents the difference of level of the two lines, being very small, is generally taken in smaller units than those by which v and r are measured. Let AC

be represented by h. Now, as the angle α is very small, in the triangle ABC, AB may be taken approximately equal to BC.

$$\text{Then, } \sin\alpha = \frac{AC}{AB} = \frac{AC}{BC} = \tan\alpha.$$

$$\text{Thus, } \tan\alpha \text{ may be taken as } \frac{h}{G}$$

$$\text{Hence, } h = G\tan\alpha = G.\frac{v^2}{gr} \qquad \cdots\cdots\cdots\cdots Eq.\ 40$$

It is needless to mention here that h and G must be taken in the same units.

Illus. Ex. 46. *What is the inclination of a curved banked track if a motorist can drive his car safely with a velocity of 30 miles per hour? The radius of the circular path is 1000 feet.*

If the inclination be α,

$$\tan\alpha = \frac{v^2}{gr} = \frac{44\times 44}{32\ 2\times 1000} = .06$$

$$\therefore\ \alpha = 3° - 26'.$$

Illus. Ex. 47. *Find the radius of the curve in a circular railway track if the outer rail of the track of a 4 ft. 8½ in. gauge is elevated by 2 in. The condition is that up to the limit of the speed of 30 miles per hour there is no thrust on the flanges of the wheels.*

$h = 2$ in.

$G = 56.5$ in

$v = 30$ m.p.hr. $= 44$ ft. per second.

Now, putting the values in *Eq.* 40.

$$2 = 56.5\times\frac{44\times 44}{32\ 2\times r}\cdot\quad \therefore\ r = \frac{56.5\times 44\times 44}{32\ 2\times 2} = 1700 \text{ feet}$$

Illus. Ex. 48. *What is the centrifugal thrust on the outer rail of a horizontal circular track of 1000 feet radius when a locomotive weighing 100 tons has a speed of 15 miles per hour?*

$v = 15$ m.p.hr. $= 22$ ft. per sec.

$r = 1000$ feet. $\qquad P = \dfrac{w}{g}\cdot\dfrac{v^2}{r}$, where P is the centrifugal thrust.

$w = 100$ tons $= 100\times 2240$ lbs.

Substituting the values of w, v and r in the equation,

$$P = \frac{100 \times 2240}{32.2} \cdot \frac{22 \times 22}{1000} = 3365 \text{ lbs. wt.}$$

101. Conical Pendulum. A heavy particle suspended from a fixed point with a fine, weightless, inextensible but perfectly elastic cord

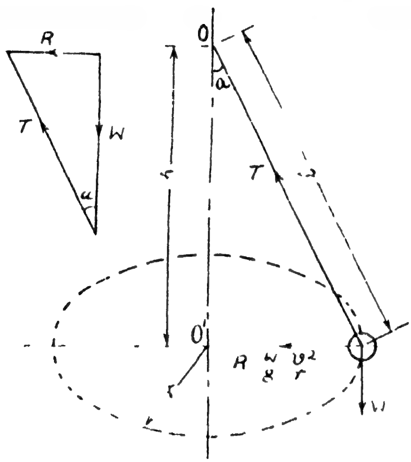


FIG. 53

constitutes a *Conical Pendulum*. Practically a spherical metallic mass of small diameter and a fine twisted silk cord are used to form the pendulum. The ball rotates in a horizontal circle about a point O' vertically below the point of suspension O (Fig. 53). Suppose the suspended mass weighing W is fastened to one end of the string, the other end of which is attached to a fixed point O. When the mass rotates in a horizontal circle about O' it keeps the string tight. The tension of the string and the weight of the ball produces the necessary amount of the accelerating force on the mass towards the centre O'. If,

T represents the tension in the string in pounds weight
ω „ the constant angular velocity in radians per second
r „ the radius of the circular path in feet
l „ the length of the pendulum from the point of suspension to the centre of the metallic bob in feet

h represents *the* height of the pendulum $(OO)'$ in feet

a „ the angle of inclination of the string with the vertical line OO' in degrees

g „ the acceleration due to gravity in feet second unit, then, from the vector diagram (Fig. 53) R, the resultant of W and T, is equal to

$$\frac{W}{g} \cdot \omega^2 r \text{ (gravitational unit)}$$

$$\text{and } \tan \alpha = \frac{W}{g} \omega^2 r \div W = \frac{\omega^2 r}{g}.$$

Again, from the diagram,

OO', *i.e.*, h (which is equal to $l \cos \alpha$)

$$= r \div \frac{\omega^2 r}{g} = \frac{g}{\omega^2} \quad \cdots\cdots\cdots\cdot Eq. 41$$

Hence, the height of the conical pendulum is dependent on the angular velocity of the bob about O' and it is found to be inversely proportional to the square of that velocity. This principle of the conical pendulum is utilised for many practical purposes, specially in governing the steam-ports of a steam engine.

From the above relation between the height and the angular velocity,

$$\omega^2 = \frac{g}{h}. \text{ Therefore, } \omega = \sqrt{\frac{g}{h}} \text{ radians per second.} \qquad Eq. 42$$

The time for one complete revolution of the bob must be equal to

$$\frac{2\pi}{\omega} \text{ or } 2\pi \sqrt{\frac{h}{g}} \text{ seconds.} \quad \cdots \quad \cdots Eq. 43$$

And if N be the number of revolutions per minute, then

$$N = 60 \div 2\omega \sqrt{\frac{h}{g}} = \frac{30}{\pi} \sqrt{\frac{g}{h}} \quad \cdots\cdots\cdots\cdots Eq. 44$$

Hence it is found that the angular velocity is inversely proportional to the square root of h; the time for one complete rotation is directly proportional to the square root of h and the number of revolutions per minute is again inversely proportional to the square root of h. It is to be noted that the difference between the magnitudes of ω and N is only with the constant factors in the equations. In case of angular velocity it is only $\sqrt{g}$, while in the other it is

$$\frac{30}{\pi}\sqrt{g}$$

Again, $\frac{W}{T} = \cos\alpha$

Or $T = \frac{W}{\cos\alpha}$ ....*Eq.* 45

Illus. Ex. 49. *What percentage of change will occur in the height of a conical pendulum making 100 rotations per minute when the speed decreases by 1.5 per cent?*

$h = \frac{g}{\omega^2}$ $\therefore$ When N is 100, the height $= \dfrac{g}{\left(\dfrac{100 \times 2\pi}{60}\right)^2} = .2941$ foot.

When the rotations decrease by 1.5 per cent, *i.e.*, when $N = 98.5$

the height becomes $= \dfrac{g}{\left(\dfrac{98.5 \times 2\pi}{60}\right)^2} = .3031$ foot.

$\therefore$ the increment in the height $= (.3031 - .2941) = .009$ foot

$\therefore$ the percentage of increment $= \frac{.009}{.2941} \times 100 = 3.06$

Illus. Ex. 50. *In a conical pendulum the weight of the bob is 4 lbs. and the height of the pendulum is 8 in. Find the linear speed of the bob if the radius of the circle described by it be 5 in. If an additional force of 1 lb. be applied downwards vertically on the bob, what should be the angular speed to maintain the same height?*

$h = \frac{g}{\omega^2}$, but again $\omega = \frac{v}{r}$. $\therefore h = \frac{g\,r^2}{v^2}$ or $v = \sqrt{\frac{g\,r^2}{h}}$.

Now, substituting the values of g, r and h in the equation for v,

$$v = \sqrt{\frac{32.2 \times 25}{12 \times 8}} = 2.89 \text{ feet per second.}$$

In the second case, $\tan \alpha = \dfrac{r}{h}$, but again,

$$\tan \alpha = \frac{W}{g}\,\omega^2 r \div (W + 1) = \frac{W}{W+1} \cdot \frac{\omega^2 r}{g}$$

$$\therefore\ h = r \div \tan \alpha = \frac{W+1}{W} \cdot \frac{g}{\omega^2}$$

Now, substituting the values of h, W and g in the above equation,

$$\frac{8}{12} = \frac{5}{4} \cdot \frac{32.2}{\omega^2} \quad \text{or,}\ \omega^2 = \frac{5 \times 32.2 \times 12}{8 \times 4} = 60.375.$$

$\therefore\quad \omega = 7.95$ radians per second.

FORCE PRODUCING CIRCULAR MOTION IN A VERTICAL PLANE

102. The necessary centripetal force can be created generally in three ways—(1) directly by the tension of a cord, (2) by the inward pressure of a circular track and (3) by the guidance of a circular

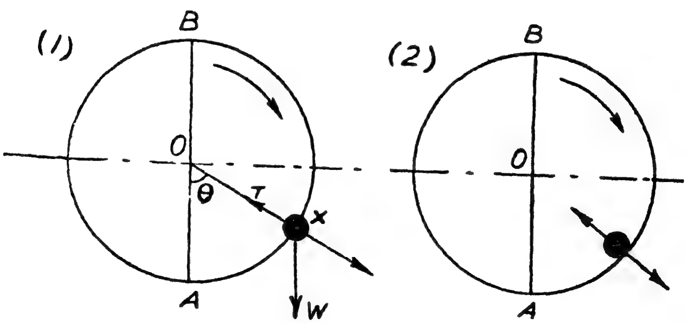


FIG. 54

tubular track. In the first two cases, let v_1 and v_2 be the velocities of a particle moving along a vertical circular path at A and B respectively, the lowest and the highest points in the path (Fig. 54—1 & 2). If W be the weight of the particle and r be the radius of the path, the

centripetal force changes from the maximum value, $\frac{W}{g} \cdot \frac{v_1^2}{r} + W$

at A to the least value, $\frac{W}{g} \cdot \frac{v_2^2}{r} - W$ at B

Therefore, for a continuous circular motion it is necessary that the least force at B towards the centre of the path must be greater than zero. That is to say,

$$\frac{W}{g} \cdot \frac{v_2^2}{r} > W \text{ or } v_2^2 > gr \qquad \text{But } v_1^2 - v_2^2 = 2gh$$

Therefore, $v_2^2 = v_1^2 - 4gr$ or $v_1^2 - 4gr > gr$

$$\text{or, } v_1^2 > 5gr \qquad \text{or, } v_1 > \sqrt{2g \cdot \frac{5}{2} r}.$$

Hence for a continuous circular motion in the vertical plane the velocity of the particle at A should be more than the velocity that the particle would acquire if it were allowed to fall freely from a height $\frac{5}{2}r$.

This principle is utilised in the making of centrifugal railways Popular application of the principle is found in circus parties, where a man with a wheeled wagon takes his start from a height which is greater than $\frac{5}{2}r$ along an inclined track and revolves once in the continuous circular path of radius r and finishes his performance.

However, it is proved that the velocity at B can never be allowed to be less than $\sqrt{gr}$ for a continuous circular motion If at any point in the vertical circular path the velocity be less than a quantity which being gradually diminished by the action of gravity becomes equal to or less than $\sqrt{gr}$ at the point B, the particle ceases to move in the circular path, because, the tension in the cord or the reaction of the track, as the case may be, falls short of the necessary amount from that point.

Thus, to satisfy the condition, the height from which the body should take the start, $h > \frac{5}{2}r$ ...... *Eq.* 46

It is to be mentioned here that the tension in the cord or the normal pressure in a circular track is equal to $\frac{W}{g} \cdot \frac{v_x^2}{r} + W \cos \theta$, where x is any point on the path and θ is the angle made by the radius OA (Fig. 54-1) with the direction of the normal tension or pressure.

The case of tubular track is not similar to the previous cases. In this case the motion is directed by the surface of the track as guide (Fig. 55) and therefore, the particle will remain moving in the

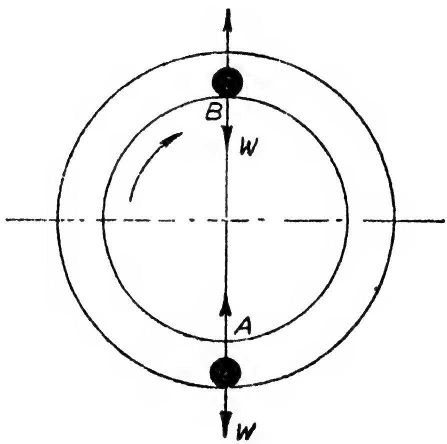


FIG. 55

track until and unless v_2 be equal to zero. Hence, for a continuous circular motion, in this case, the necessary condition is that v_2 must be greater than zero. But, $v_2^2 = v_1^2 - 4gr$. Therefore, to satisfy the condition $v_1^2 > 4gr$ or $v_1 > \sqrt{2g. 2r}$, *i.e.*, the velocity at A will be such that it falls from a height more than the diameter of the circular tubular track.

Thus, $v_1 > \sqrt{2gd}$, ...*Eq.* 47

where d is the diameter of the circular track.

Illus. Ex. 51. *A metallic ball weighing ¼ lb. is rotated in a vertical plane with a string 2.5 ft. long. If the velocity of the ball be such that it can just complete the circle, find the linear speed of the ball and the tension in the string—(1) at the highest position, (2) at the lowest position and (3) at a position making an angle of 90° with the vertical diameter at the centre.*

Just to complete the circle the centrifugal force at the highest point must be equal to the weight of the body.

$$\therefore \tfrac{1}{8} = \frac{1}{8 \times 32.2} \cdot \frac{v_2^2}{2.5} \quad \text{or,} \quad v_2 = \sqrt{2.5 \times 32.2} = 8.97 \text{ feet per second}$$

Under the circumstances the velocity at the lowest point will be such that,

$$v_1 = \sqrt{5 . g . r} = \sqrt{5 \times 32.2 \times 2.5} = 20 \text{ feet per second.}$$

Tension in the string is equal to the weight + the centrifugal force

$$= \tfrac{1}{8} + \frac{1}{8 \times 32.2} \times \frac{5 \times 32.2 \times 2.5}{2.5} = .75 \text{ lb} \quad (i.e., \text{ six times the weight})$$

Now, in the last case, if v' be the velocity at the point,

$$v'^2 = v_2^2 + 2 \times 32.2 \times 2.5 = 32.2 \times 2.5\,(1 + 2) = 3 \times 2.5 \times 32.2$$

$$\therefore v' = 15.5 \text{ feet per second}$$

$$\text{and the tension} = \frac{1}{8 \times 32.2} \times \frac{(15.5)^2}{2.5}$$

$$= .375 \text{ lb} \quad (i.e., \text{ three times the weight})$$

FORCE PRODUCING SIMPLE HARMONIC MOTION

103. If a particle weighing w moves in a simple harmonic motion, the force required to produce an acceleration of magnitude, $k.x$, is equal to $\frac{w}{g}\,k.x$ (gravitational unit).

Thus, if P represents the force, $P = \frac{w}{g}\,k.x$. But $k = \frac{4\pi^2}{t^2}$

$$\text{therefore, } P = \frac{w}{g} \cdot \frac{4\pi^2}{t^2} \cdot x \quad \ldots \qquad Eq.\ 48$$

From the equation it is found that the force, like acceleration, varies directly with the distance.

The diagram (Fig. 56) shows the acceleration-space curve and the force-space curve. It is found that the acceleration is maximum at

A, it then gradually diminishes to zero at O and increases again to the maximum value at B in the opposite direction. Suppose the

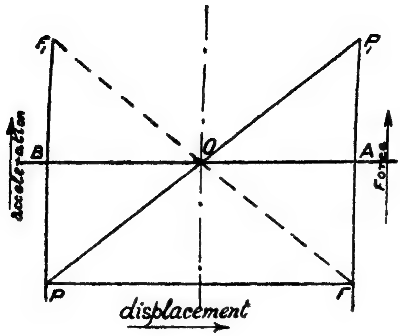


FIG. 56

acceleration-space curve is FOF_1; then, because the acceleration is directly proportional to the force producing it, the force-space curve also will be a straight line like POP_1.

104. Simple Pendulum. A heavy particle, suspended from a fixed point with a fine, weightless, inextensible but perfectly flexible cord constitutes a simple pendulum.

A heavy particle and a thread as mentioned above are both practically impossible things. Actually a spherical metallic bob of small diameter, varying from 2 to 3 c.m. and a fine twisted silk cord are used. The length of the pendulum is counted from the point of suspension to the centre of the bob, as if the total mass is concentrated at that centre.

Suppose the bob of the pendulum is at an extreme end of the amplitude (Fig. 57). The weight of the bob W acts vertically downwards and the tension T of the cord acts along it as shown in the diagram. These are the only two forces that act in the system to produce the motion of oscillation. This motion is found on experiment to follow approximately the laws of simple harmonic motion, when the angle of oscillation is small. If the angle θ is not greater than $5°$, the motion of the bob corroborates with simple

harmonic motion up to 3 places of decimals. The magnitude of the resultant of the two forces acting in the system,

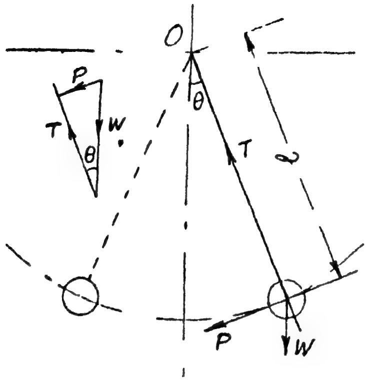


FIG 57

$$P = \frac{W}{g} \cdot \frac{4\pi^2}{t^2} \cdot x,$$ where x, which is equal to $l \sin\theta$

(l being the length of the pendulum and θ, the angle between the vertical and immediate positions of the pendulum), is the horizontal distance of the centre of the bob from the neutral position of the system. P always acts tangentially to the path of oscillation

From the vector triangle in the diagram it is found that, $P = W \sin\theta$ Therefore,

$$\text{or,}\quad P = \frac{W}{g} \cdot \frac{4\pi^2}{t^2}\, l \sin\theta = W \sin\theta$$

$$\text{or}\quad t^2 = \frac{4\pi^2 l}{g}, \text{ whence, } t = 2\pi\sqrt{\frac{l}{g}} \qquad \textit{Eq } 49$$

If n be the number of complete oscillations per second, then,

$$n = \frac{1}{t} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \qquad \textit{Eq. } 50$$

Thus, periodic time is directly proportional to the square root of l, whereas the number of complete oscillations per second is inversely proportional to the square root of l.

It is to be marked that x, the distance, is measured as $l\,\theta$. Hence, in case of motion like that of a pendulum bob x is always to be measured along the curved path unlike the case of motion of a helical spring, where x is the straight distance of a moving particle from the neutral position.

105. Galileo proved experimentally that the amplitude of oscillation of a simple pendulum being within a reasonable limit, the time of swing, *i.e.*, the time for a complete oscillation remained constant. The value is found to corroborate with the result of $2\pi\sqrt{\frac{l}{g}}$

Thus the mathematical deduction is corroborated by the experimental results to show that the motion of a simple pendulum is simple harmonic in nature.

106. It is also experimentally proved that in a given place the time of swing of a simple pendulum is independent of the material of which the bob of the pendulum is composed.

107. Seconds Pendulum. A pendulum which takes one second to travel from one extreme end to the other of the amplitude, *i.e.*, half of a complete oscillation is called a seconds pendulum.

From the equation, $t = 2\pi\sqrt{\frac{l}{g}}$, then, $1 = \pi\sqrt{\frac{l}{g}}$

where l is the length of the pendulum. (1 second being the time for half oscillation).

$$\therefore l = \frac{g}{\pi^2} \qquad \cdots \textit{Eq. } 51$$

Thus, with the variation of the value of g at different places l also will vary.

In London, then, the length of a seconds pendulum should be approximately equal to $\frac{32.2}{(3.14)^2} = 3.264$ feet or 39.168 in. In C.G.S. system the value of l becomes approximately equal to 99.5 c.m,

Illus. Ex. 52. *Find the length of a pendulum which makes 59 beats in a minute. Take the value of g as 32·2 ft per sec per sec.*

The time for one beat $= \dfrac{60}{59}$ seconds. Hence, $\dfrac{60}{50} = \pi\sqrt{\dfrac{l}{g}}$

$$\therefore l = \frac{60 \times 60 \times 32.2 \times 12}{59 \times 59 \times 3.14 \times 3.14} = 40.54 \text{ in}$$

Illus. Ex. 53. *From a Laboratory experiment it is found that a pendulum, 1 metre in length, makes 314 beats in a period of 5 minutes; find the value of g of the place.*

$t' = \pi\sqrt{\dfrac{l}{g}}$, where t' is the time for one beat or half oscillation.

$\therefore g = \dfrac{\pi^2 l}{t'^2}$ But, here, $t' = \dfrac{5 \times 60}{314}$ sec

$$\therefore g = \frac{\pi \times \pi \times 1 \times 314 \times 314}{5 \times 60 \times 5 \times 60} = 10.65 \text{ metres per sec per sec}$$

Illus. Ex. 54. *A faulty seconds pendulum, where g is equal to 32·2 ft per second per second loses 10 seconds per day, find what correction is to be made in its length so that it keeps a correct time.*

1 day = 86400 seconds $\therefore$ the number of beats in a day = 86390 (in the given pendulum)

$\therefore$ Time for each beat $= \dfrac{86400}{86390} = \dfrac{8640}{8639}$ seconds.

$$\therefore \frac{8640}{8639} = \pi\sqrt{\frac{l}{g}} \text{ or, } l = \left(\frac{8640}{8639}\right)^2 \times \frac{g}{\pi^2}$$

But correct length of a seconds pendulum $= \dfrac{g}{\pi^2}$. Thus, it is found that the length of the given pendulum is longer than what it should be.

Therefore, for keeping correct time the length should be shortened by

$$\frac{g}{\pi^2}\left\{\left(\frac{8640}{8639}\right)^2 - 1\right\} \text{ ft} = \frac{32.2}{3.14 \times 3.14}\left\{\left(\frac{8640}{8639}\right)^2 - 1\right\} \times 12 \text{ in}$$

$$= .009 \text{ in.}$$

108. The value of g can be found out accurately with the help of a pendulum.

The pendulum with which the experiment should be done, is set swinging before a clock whose pendulum is known to beat true seconds. *The instant when the two pendulums coincide vertically* and have motions in the same sense, is recorded. The motions are observed for a period till the two pendulums have again the similar positions and motions with which the recording was started. From the number of oscillations of the experimental pendulum and that of the other one which records the true time in seconds, the periodic time of the experimental pendulum can be found out. Then, with the help of the relation, $t = 2\pi\sqrt{\frac{l}{g}}$ the value of g can easily be calculated.

109. Helical Spring. Experimentally it is found that the vibration of a helical spring obeys the condition of simple harmonic motion. In this case the elongation and the shortening of the spring are directly proportional to the load applied up to a definite limit. If e be the force that is required to disturb the length by one unit, then ex is equal to the force in action for a deflection of x units from the mid-position. e is called the *stiffness of the spring.*

Let W be the weight of the mass in motion due to the vibration of the spring. Now, k being the acceleration at unit distance,

$$k = \frac{\text{accelerating force at unit distance}}{\text{mass}}$$

$$= e \div \frac{W}{g} = \frac{e.g}{W}$$

and the periodic time,

$$t = \frac{2\pi}{\sqrt{k}} = 2\pi\sqrt{\frac{W}{eg}} = 2\pi\sqrt{\frac{m}{e}} \qquad \text{........} Eq.\ 52$$

where m is the mass of the body in motion.

Illus. Ex. 55. *A light helical spring deflects $\frac{1}{8}$ of an inch per pound of load. Find the load required to set the spring in oscillation of 200 vibrations per minute.*

$$e = 8 \times 12 = 96 \text{ lbs.}$$

$$t = 2\pi\sqrt{\frac{W}{ge}} \quad \therefore \quad W = \frac{t^2 \times g \times e}{4\pi^2}, \text{ but' } t = \frac{60}{n}$$

$$\therefore \quad W = \frac{60^2 \times 32.2 \times 96}{200^2 \times 4 \times (3.14)^2} = 7.05 \text{ lbs.}$$

Illus Ex 56. *A beam deflects 1·125 in under a load of 1 ton at the middle of the span Neglecting the weight of the beam find the period of vibration*

$$c = \frac{(12 \times 2240)}{1{\cdot}125}$$

$$t = 2\pi\sqrt{\frac{m}{c}} = 2\pi\sqrt{\frac{2240 \times 1{\cdot}125}{32{\cdot}2 \times 12 \times 2240}} = {\cdot}338 \text{ seconds}$$

Illus Ex. 57. *In a steam engine the length of the crank is 15 in and the connecting rod is big enough to allow the motion of the piston to be considered as simple harmonic If the piston with the reciprocating parts weighs 350 lbs and if the crank rotates 150 times per minute, find the accelerating forces when the crank, in its forward motion, makes an angle of 60° with the inner dead centre position, and also when the piston is at a distance of 9 in from the outer end of the stroke What are the velocities of the piston at those points?*

$$\omega = \frac{(150 \times 2\pi)}{60} = 5\pi \text{ radians per second}$$

$$f = a\,\omega^2 \cos \omega t$$

$\therefore$ in the first case it is equal to,

$$\frac{15}{12}(5\pi)^2 \cos 60$$

$\therefore$ the accelerating force is equal to,

$$\frac{350}{32{\cdot}2} \times \frac{15}{12}(5\pi)^2 \times {\cdot}5$$

$= 1675$ lbs, the direction of the force is towards the mid centre

In the second case, the position of the piston is 9 in from the outer end of the stroke, *i e*, $(15 - 9) = 6$ in away from the mid-centre

The value of $\cos \omega t$ here is $-\dfrac{6}{15} = -{\cdot}4$

$$\text{the accelerating force} = \frac{350}{32{\cdot}2} \times \frac{15}{12}(5\pi)^2 \times -{\cdot}4$$

$$= -1340 \text{ lbs}$$

From the negative sign it is understood that the direction is towards the mid centre

The velocities are $1{\cdot}25 \times 5\pi \times {\cdot}5$ and $1{\cdot}25 \times 5\pi \times -{\cdot}4$

(because $v = a\omega \sin \omega t$)

i e, $= 9{\cdot}8$ ft per second and $-7{\cdot}86$ ft per second respectively

The signs indicate that the first one is towards the mid-centre and the second one is away from the mid-centre.

Illus. Ex. 58. *Find the time of a complete oscillation of a column of liquid which occupies a length of 2 feet of a U-tube* (shown in Fig. 58).

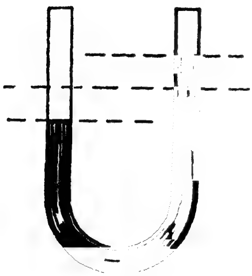


FIG. 58

$$t = 2\pi \sqrt{\frac{W}{ge}}$$

Now, to move the liquid from the neutral position by 1 foot in the tube a force equal to the weight of the column is required, *i.e.*, if W be the weight of the column, $e = W$.

Therefore, $t = 2\pi \sqrt{\dfrac{W}{g\,W}} = 2\pi \sqrt{\dfrac{1}{32.2}}$

$= 1.107$ seconds.

IMPACT—AN INSTANCE OF IMPULSIVE FORCE

Smooth Spherical Bodies

Here is a deviation from the point of consideration of particles. Here, actual bodies—smooth and spherical—are treated as it will be more advantageous to discuss the thing at this place.

110. *The exertion of force is made through impact, attraction or repulsion.* Generally, excepting in cases of impact, the exertion is effected through some medium. When there is an application of force through a medium, the medium is said to be in a state of stress.

111. Stress. When two equal and opposite forces act on a piece of material and change the original shape of the piece, the piece is said to be in a *state of stress*.

112. When the direction of motion in the two bodies is along the common normal at the point of contact the bodies are said to *impinge directly* and when either one or both of the bodies have got motion not in a line with the common normal they are said to *impinge obliquely*. The line of common normal is called the *line* of impact. In case of spherical bodies the straight line joining the two centres is the *line of impact*.

113. From the results of experiments, Newton deduced the law that, if a body impinges directly against another body, their relative

velocity *after the impact bears a constant ratio with the relative* velocity before the impact, and the two velocities are in opposite directions.

But if the bodies impinge obliquely then also the components of their relative velocities along the common normal after and before the impact are in the same constant ratio and are of opposite signs.

If u and u' be the initial velocities of the two bodies in case of direct strike or if they be the components of those velocities along the common normal in case of an oblique strike before the impact and if v and v' be their corresponding values after the impact, then,

$$(v - v') \div (u - u') = \text{a constant quantity.}$$

114. This constant quantity depends on the composing materials of the bodies, but is independent of the quantity of masses. It is always found to be a negative fraction. In case where u is greater than u', v is found to be less than v'. Thus, though $(u - u')$ is a positive value, $(v - v')$ will be negative and the ratio is found to be always less than unity and is represented by a letter e; and

$$\text{therefore, } \frac{v - v'}{u - u'} = -e. \qquad \cdots\cdots\cdots\cdots Eq.\ 53$$

This quantity 'e' is called the *Coefficient of restitution* or *Modulus of elasticity*. The above relation can be utilised for solving many problems regarding the impact of bodies.

115. Elasticity. Impact produces a certain amount of deformation in the physical shape of the bodies as is shown in the diagram (Fig. 59, drawn in an exaggerated scale). But it is found that when the greatest compression is complete the bodies begin to come back to their original shapes. This property of coming back to the original shape is called the *elasticity* of the materials of the bodies.

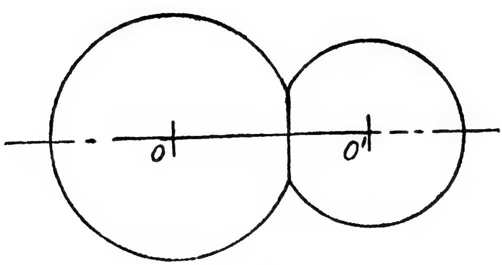


FIG. 59

The velocity of rebound depends on this property of the materials. If a rubber ball be dropped on a floor, the ball after the impact will rise to a certain height which depends on the initial velocity of the ball, *i.e.*, the velocity just after the impact. This initial velocity after the impact is called the *velocity of rebound*. The bodies which have greater velocity of rebound are called to be more elastic.

116. Every body is more or less elastic. When the value of e is equal to 1, the bodies are said to be perfectly elastic and where the value of e is zero the two bodies adhere and move as a single mass with a common velocity and the bodies are said to be inelastic. Accurate experiments show that the value of e does not *absolutely* remain constant. However, taking an approximate value for e, problems can be solved out with the help of the above equation without any appreciable difference.

117. In case of oblique impact, only the components of the velocities along the line of impact must be considered to be changed, because the components along the direction at right angles to the line of impact remain unaltered owing to the smoothness of the spherical surfaces.

118. The angle at which a body strikes another body is called the *angle of incidence*, and the angle at which the velocity of rebound directs is called the *angle of reflexion*. θ (Fig. 60) is the angle of incidence, and λ is the angle of reflexion.

119. Different problems on impact of two bodies:

1. (*a*) *Find the velocity of rebound if a body be dropped on a fixed plane and if the impact be a direct one.*

By the term 'fixed plane' it is meant that the body has a very big mass and dimension in comparison with those of the dropped body, and it is at rest. Now,

$$\frac{v - v'}{u - u'} = -e,$$

$$\therefore \quad v - v' = -e(u - u'). \qquad \cdots\cdots Eq.\ 54$$

Thus, if u' be zero, and because the mass of the fixed plane is taken to be very big, v' is approximately equal to zero. Then,

$$v = -eu. \qquad \cdots\cdots\cdots Eq.\ 55$$

(*b*) When the impact is an oblique one, let u and v be the velocities of the body before and after the impact, in the direction as shown in the diagram (Fig. 60).

The body and the plane being smooth there is no force in a direction parallel to the plane, hence the component of the velocity of the body in that direction before and after impact will remain unaltered.

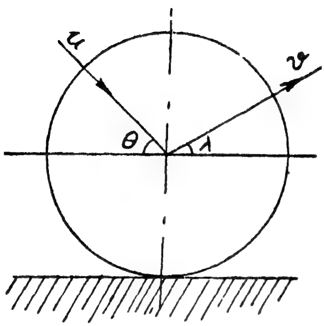


Fig. 60

Therefore, $v \cos \lambda = u \cos \theta$ ...(*i*)

Again from the previous case

$v \sin \lambda = - e \times - u \sin \theta = e.u. \sin \theta$

($v \sin \lambda$ and $u \sin \theta$ are in opposite direction) ... ...(*ii*)

From (*i*) and (*ii*), squaring both the sides and then adding,

$v = u\sqrt{\cos^2 \theta + e^2 \sin^2 \theta}$ ···*Eq.* 56

and dividing (*ii*) by (*i*)

$\tan \lambda = e. \tan \theta$ ··· *Eq.* 57

Thus the magnitude and direction of the velocity of rebound are obtained.

2. (*a*) *Find the velocities of two smooth spherical bodies after the impact if they impinge directly.*

If u and u' be the velocities before the impact and v and v' be those after the impact, then by the principle of conservation of momentum

$$m v + m' v' = m u + m' u', \quad \ldots \quad \ldots \quad (i)$$

where m and m' are the masses respectively.

$$\text{Again,} \quad v - v' = - e (u - u') \quad \ldots \quad \ldots \quad (ii)$$

Multiplying (*ii*) by m' and adding the result with (*i*)

$$(m + m').v = (m - em') u + m' u' (1 + e) \quad \ldots \quad (iii)$$

Similarly, $(m + m') v' = (m' - em) u' + mu (1 + e)$ ... (*iv*)

These two equations give the values of v and v' respectively in terms of u, u', m, m', and e.

Now subtracting $(m + m') u$ from both the sides of the equation (*iii*)

$$(m + m') \; (v - u)$$

$$= (m - em') u + m' u' (1 + e) - (m + m') u$$

$$\text{Hence,} \quad (v - u) = \frac{m' (1 + e)(u' - u)}{(m + m')}$$

$$\text{Similarly,} \quad v' - u' = \frac{m' (1 + e)(u - u')}{(m + m')}$$

Therefore, the impulses of the blow due to the impact on two bodies respectively are equal to $m\,(v - u)$ and $m'\,(v' - u')$, which are equal in magnitude but opposite in direction.

$$\text{Hence, the } \textit{impulse} = \text{either} \quad \frac{m\,m'}{m + m'}\,(1 + e)\,(u' - u)$$

$$\text{or} \quad \frac{m\,m'}{m + m'}\,(1 + e)\,(u - u') \quad \cdots\cdots \; \textit{Eq.}\ 58$$

Cor. When e is equal to zero, *i.e.*, when the bodies are perfectly inelastic, the two bodies will adhere and move with a common velocity. The magnitude of this common velocity can be found out directly from the equation, $m\,u + m'\,u' = (m + m')\,v$, where v is the required velocity, or, from the equation (m) or (w), where $v = v'$ taking the value of e as zero.

$$\text{Thus,} \quad v = \frac{(mu + m'\,u')}{(m + m')}$$

(b) When the two bodies impinge obliquely, suppose the two balls are running with initial velocities u and u' respectively in the

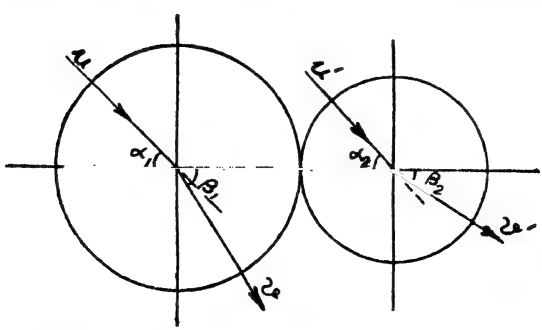


FIG. 61

directions as shown in the diagram (Fig. 61). If v and v' be the final velocities of the two balls after the impact in the directions

given in the diagram, the components of u, u', v and v' along the line of impact are $u \cos \alpha_1$, $u' \cos \alpha_2$, $v \cos \beta_1$ and $v' \cos \beta_2$ and the components along the direction at right angles to the line of impact are, $u \sin \alpha_1$, $u' \sin \alpha_2$, $v \sin \beta_1$ and $v' \sin \beta_2$ respectively. Then because there is no force acting along the direction at right angles to the line of impact at the point of contact due to the smoothness of the surfaces, the components of the velocities along that direction will remain unaltered,

$$i.e., \quad v \sin \beta_1 = u \sin \alpha_1 \qquad \ldots \qquad (i)$$

$$\text{and} \quad v' \sin \beta_2 = u' \sin \alpha_2 \qquad \ldots \qquad (ii)$$

And the total quantity of momentum before and after the impact along the line of impact will remain constant, therefore,

$$m\,v \cos \beta_1 + m'\,v' \cos \beta_2 = m\,u \cos \alpha_1 + m'\,u' \cos \alpha_2 \qquad \ldots \qquad (iii)$$

Again, (articles 114 and 117)

$$v \cos \beta_1 - v' \cos \beta_2 = -\,e\,(u \cos \alpha_1 - u' \cos \alpha_2) \qquad \ldots \qquad (iv)$$

Multiplying (iv) by m' and adding with (iii),

$$v \cos \beta_1 = \frac{m'(1+e)\,u' \cos \alpha_2 + (m - em')\,u \cos \alpha_1}{m + m'} \qquad \ldots \qquad (v)$$

In a similar way,

$$v' \cos \beta_2 = \frac{m(1+e)\,u \cos \alpha_1 + (m' - em)\,u' \cos \alpha_2}{m + m'} \qquad \ldots \qquad (vi)$$

By squaring and adding (i) and (v), v^2 is obtained and similarly from (ii) and (vi) v'^2 is found out. One can easily get the values of $\tan \beta_1$ and $\tan \beta_2$ by dividing (i) by (v) and (ii) by (vi) respectively. Thus the magnitudes and directions of the velocities are found out.

The impulse in this case is $m\,(v \cos \beta_1 - u \cos \alpha_1)$.

Proceeding in the way of the last problem this is equal to

$$\frac{m\,m'}{m + m'}\,(1 + e)\,(u' \cos \alpha_2 - u \cos \alpha_1) \qquad \cdots\cdots\cdot Eq.\ 59$$

Cor. *If u' be zero then β_2 becomes zero.*

That is, the second body will move along the line of impact; which can also be understood clearly from the fact that due to the smoothness of the surfaces there is no force acting along the direction at right angles to the line of impact and the action of the blow due to the impact will be along the line of impact.

120. Condition of bodies during the Impact. The period 't' for which a force acts to create the total change of momentum can be divided into two parts, the period of compression t_1 and the period for restitution t_2. Then if A be the total change in momentum or the total impulse of the force P (say), and if A_1 and A_2 be the changes in momentum in t_1 and t_2, *i.e.*, the impulses of the force P during t_1 and t_2,

$$\text{then } A = P.t$$

$$A_1 = P.t_1$$

$$\text{and } A_2 = P.t_2$$

$$\text{But } A = A_1 + A_2$$

$$i.e., \quad P.t = Pt_1 + Pt_2$$

Now, if u and u' be the initial velocities of the two moving bodies of masses m and m' respectively and v_1 be their common velocity when the compression is going on,

$$m\,u + m'\,u' = (m + m').\ v_1$$

$$\text{or} \quad m\,u - m\,v_1 = m'\,v_1 - m'\,u'$$

$$\text{or} \quad m\,(u - v_1) = m'\,(v_1 - u') = A_1$$

$$\text{Thus} \quad \frac{A_1}{m} + \frac{A_1}{m'} = u - u' \qquad \ldots \qquad (i)$$

Similarly, when the restitution takes place,

$$A_2 = m\,(v_1 - v) = m'\,(v' - v_1),$$

where v and v' are the velocities after the restitution is complete.

$$\text{Therefore,} \quad \frac{A_2}{m} + \frac{A_2}{m'} = v' - v \qquad \ldots \qquad (ii)$$

From (*i*) and (*ii*) then,

$$\frac{A_2}{A_1} = \frac{(v' - v)}{(u - u')} = e$$

$$i.e.,\ A_2 = e\,A_1 \qquad \cdot \quad Eq.\ 60$$

It is to be marked in this case that during the compression period the two bodies move as if they form a single body with a common velocity but after the compression is complete the two bodies get rid of the deformation through the property of elasticity and gradually are detached and run with two different velocities of their own.

Illus Ex. 59. *A rubber ball is thrown vertically upwards in a room and just touches the ceiling It then drops on the floor and is found that after rebounding twice it rises up $\frac{1}{4}$ of the height of the room Find the coefficient of restitution.*

The velocity of rebound after first fall $= e\sqrt{2gh}$, where h is the height of the room

$$\therefore \text{ the height it can rise due to this velocity} = e^2h$$

The velocity of rebound after the second impact then $= e\sqrt{2ge^2h}$ and the height through which the ball can rise with this velocity is equal to e^4h.

$$\text{Therefore, } e^4h = \tfrac{1}{4}h \text{ or } e = \sqrt[4]{\tfrac{1}{4}} = .707$$

Illus Ex. 60. *If a ball moving with a velocity of 10 ft per second strikes a smooth plane at an angle of 30°, find the magnitude and direction of the velocity after the impact, where the coefficient of restitution is $\frac{1}{2}$.*

If v be the magnitude of the velocity,

$$v = 10\sqrt{\cos^2 30 + \tfrac{1}{4}\sin^2 30} \qquad (Eq\ 56)$$

$$= 9 \text{ ft per second.}$$

If λ be the direction of the velocity with the plane,

$$\tan\lambda = \tfrac{1}{2}\tan 30 = .2887$$

$$\therefore \lambda = 16^\circ - 6' \text{ approximately}$$

Illus Ex. 61. *Compare the masses of two balls, if one overtakes the other along the same path, strikes and comes to rest after the impact. The velocity of the swifter one is 7 times than that of the other and $e = \frac{3}{4}$.*

If m and m' be the masses and v be the velocity of the mass m after the impact, and if m be swifter than m' and their velocities are u and u' respectively,

$$v = O = \frac{(m - \frac{3}{4} m')\, 7\, u' + m'\, u' \,(1 + \frac{3}{4})}{m + m}$$

Art 119—Prob 2 a—*Eq* (*iii*)

$$= 7\, m\, u' - \frac{3}{4}\; 7\, m'\, u' + \frac{7}{4}\; m'\, u'$$

or $m = \frac{1}{2} m'$,

That is, the mass of the slower ball is double the mass of the swifter one

Illus Ex 62 *A ball is dropped from a height of 19 62 metres, another of equal mass is thrown upwards along the same course at the same instant with a velocity of 39 24 metres per second They collide and come to the ground one after another If the coefficient of restitution be $\frac{3}{4}$, find when they reach the ground*

If x be the fall of the dropping ball when the collision occurs,

$x = \frac{1}{2} g t$, where t is the time to meet

Then, considering the second ball, $1962 - x = 3924\, t - \frac{1}{2} g t$

Now, adding these two equations, $1962 = 3924\, t$ or, $t = \frac{1}{2}$ second

Substituting the value of t in the first equation,

$x = \frac{1}{2} \times 981 \times \frac{1}{4} = 122\ 625$ c m

Again, if v' and v'' be the velocities of the two balls respectively after the impact, $v' = \dfrac{(m - \frac{3}{4}m')\, u + m'\, u'\, (1 + \frac{3}{4})}{m + m'}$

But as $m = m'$, $v' = \frac{1}{4}\ (u + 3u')$, where u and u' are the velocities of the balls respectively before the impact

Now, $u = 981 \times \frac{1}{2} = 490\ 5$ c m per second

and $u' = 3924 - (981 \times \frac{1}{2}) = 3433\ 5$ c m per second

But u and u' are in opposite direction, and therefore, if u is taken in the positive sense u' must be taken in the negative sense

$\therefore\ v' = \frac{1}{4}\ (490\ 5 - \overline{3 \times 3433\ 5}) = -\ 2452\ 5$ c m per second

Similarly, $v'' = -\ 490\ 5$ c m per second

That is to say, after the impact both the balls move upwards with new velocities v' and v'' starting from a height of $(1962 - 122.625) = 1839.375$ c.m.

In going up till the velocities become zero and coming down to the same level, the bodies will require (according to, $t = 2 \frac{u}{g}$)

$2 \times \frac{2452.5}{981}$ and $2 \times \frac{490.5}{981}$ or, 5 and 1 seconds respectively, and the bodies will acquire the same velocities at that point.

From that point if the dropped ball and the projected one require t' and t'' seconds respectively to fall on the ground,

$$\text{(I)} \quad 1839.375 = 2452.5\, t' + \tfrac{1}{2} \times 981 \times t'^2$$

from which $t' = .662$ second

$$\text{(II)} \quad 1839.375 = 490.5\, t'' + \tfrac{1}{2} \times 981 \times t''^2$$

from which $t'' = 1.5$ seconds.

$\therefore$ The dropped ball will reach the ground after $(.5 + 5 + .662)$ *i.e.*, 6.162 seconds from the starting and the other will reach after $(.5 + 1 + 1.5)$, *i.e.*, 3 seconds from the starting.

Illus. Ex. 63. *Two perfectly elastic balls of equal mass collide. One runs at an angle of 30° with the line of impact while the other runs with half the speed of the previous one at an angle of 60° with the line joining the centres at the time of contact. Find the magnitudes and directions of the velocities of the two balls respectively after the impact*

$$v \cos \beta_1 = \frac{m'(1+e)u' \cos \alpha_2 + (m - em')u \cos \alpha_1}{m + m'}$$

Art. 119—2 b—*Eq.* (v)

In this case masses being equal, and e being equal to 1,

$v \cos \beta_1 = u' \cos \alpha_2$, and due to the smoothness of the body $v \sin \beta_1 = u \sin \alpha_1$

$\therefore$ Squaring both the sides and adding, $v^2 = u'^2 \cos^2 \alpha_2 + u^2 \sin^2 \alpha_1$

Substituting the values of α_1 and α_2, $v^2 = \frac{5}{16} u^2$, or, $v = \frac{1}{4}\sqrt{5} \times u$

and dividing the equations, $\tan \beta_1 = \frac{u \sin \alpha_1}{u' \cos \alpha_2} = 2.$

$$\therefore \quad \beta_1 = 64°.$$

Similarly, $v' \cos \beta_2 = u \cos \alpha_1$ and $v' \sin \beta_2 = u' \sin \alpha_2$

and in the same way, $v' = \frac{1}{4}\sqrt{15} \times u$, and $\beta_2 = 27°$ approximately.

PROBLEMS

88. In a minute the velocity of a body changes from 1.5 metres per second to 4 metres per second. If the force in action be uniform and 500 dynes, find the mass. *Ans.* 120 gms.

89. A force of 15 poundals acts on a body at rest for 5 seconds after which the action is withdrawn. It is found, next, to move 100 feet in 10 seconds. What is the mass of the body? *Ans.* $7\frac{1}{2}$ lbs.

90. The weight of a composition in a railway train, exclusive of the engine is 430 tons. It attains a speed of 40 miles per hour from rest in 5 minutes on a level track. Find the pull in the tie-rod between the engine and the composition if the average resistance be 12 lbs. per ton. *Ans.* 4.916 tons.

91. A projectile weighing 100 lbs. being thrown at a speed of 1500 feet per second from a gun strikes a target with an average resistance of 1,000 tons. Find the depth of penetration. *Ans.* 1.56 feet.

92. Steam is shut off in an engine of a train running at 45 miles per hour when it rises up a slope 1 in 250. If the train be brought to rest in 1,000 yards, find the brake force applied in pounds per ton weight of the train. Neglect other resistances. *Ans.* 41.5 lbs./ton.

93. If the track resistance of a car weighing 10 tons be 25 lbs. per ton, what is the total impulse of the effective force when the engine exerts a force of 500 lbs. for 20 seconds? Find the speed of the car after that time. *Ans.* 5,000 units, 4.9 miles/hr.

94. What will be the velocity of a mass weighing 250 gms. if a force of 1,000 gms. acts on it for one minute? If the force ceases to act after that and by a sudden jerk if the velocity of the body is reduced to half, find the impulse of the jerk. *Ans.* 23.544 *k.m.*/sec.
5.887×10^6 units.

95. A bowler delivers a ball straight to the stump with a velocity of 300 feet per second. The batsman hits it to the square leg with a velocity of 400 feet per second. Find the impulse of the hit if the weight of the ball be $5\frac{1}{2}$ oz. *Ans.* 5.338 units.

96. A body is moving with constant acceleration. At an instant its velocity is 15 miles per hour and it becomes three times in ten seconds. Determine the relation between the weight of the body and the effective force. *Ans.* $P = .136$ W

97. A train weighing 322000 lbs. runs with a constant acceleration on a straight level track. The velocity changes from zero to 45 miles per hour

in 1 minute. If the resistance due to friction, air, etc. be .5% of the weight of the train, find the draw-bar pull of the train.

Ans. 12610 lbs.

98. In moving up a 1% grade a locomotive engine exerts a constant draw-bar pull of 50,000 lbs. If all the resistances combined excepting that due to the slope of the track, which may be expressed as train resistance, is 12 lbs. per ton weight of the train and if the speed changes from 30 miles per hour to 40 miles per hour in a distance of 20 miles, find the weight of the train in tons. *Ans.* 1131 tons.

99. A cyclist weighing 200 lbs. with the cycle coasts down a 2% grade with a starting speed of 10 miles per hour and running for 1,000 feet reaches a level track. If the resistance due to friction, air etc. totals to 2 lbs., find how far the cyclist can move before coming to rest.

Ans. 1349 feet.

100. An express train is coasting down an .8% grade at a speed of 60 miles per hour. The driver applies the emergency brake suddenly. The total resistance becomes 10% of the train's weight. Determine the space traversed by the train after the application of the brake. In what time will the train come to rest? *Ans.* 1308 feet, 29.7 secs.

101. A projectile weighing 1 ounce is thrown from a gun weighing 14 pounds with a velocity of 1,000 feet per second. Find the velocity of recoil.

Ans. 4.46'/sec.

102. A body weighing 12 grammes moves with a velocity of 15 c.m. per second and impinges directly on another body weighing 25 gms. moving in the same direction with a velocity of 10 c.m. per second. They coalesce and proceed to move as a single body. Find the velocity of the combined body. *Ans.* 11.62 c.m./sec.

103. Water is projected from a hose-pipe at a velocity of 22 ft. per second. What forward force is required to hold the mouth-piece in position if the cross-sectional area of the orifice in the piece be 2 square inches?

Ans. 13.05 lbs.

104. A lift with its occupants weighs 16.1 cwt. On approaching the ground-floor with a retardation of 5 ft. per second per second it is brought to rest. Find the tension in the cable. *Ans.* 2083.2 lbs.

105. A weight of 12 lbs. hangs vertically by means of a cord passing over a frictionless pulley. The other end of the cord is tied to a block of wood, weighing 15 lbs., which rests on a smooth table top. If the block offers a resistance of 3 lbs. due to a small inclination of the surface, find the acceleration created in the system and also the tension in the string.

Ans. $10.73'/\text{sec.}^2$, 8 lbs.

106. Two weights—15 gms. and 20 gms.—are suspended by a string passing over a light frictionless pulley. After being allowed to move freely for 3 seconds from the state of rest 7 gms. are taken off from the heavier weight. Find when they will come to rest again, and also find the distances moved by the two bodies respectively, from the moment of release to the moment of coming to rest again.

107. In figure 50, $W_1 = 300$ lbs., $W_2 = 700$ lbs. and the resistance to motion is 200 lbs. Compute the amount of force that should be applied on W_2 at an angle of 30° with horizontal downwards so that the system which is moving with a speed of 20 ft. per second at the instant of the application of the force may be brought to rest in a distance of 40 feet. Also, compute the tension in the cord. Due to the obliquity of the application of the force, the vertical component increases the resistance by one-sixth of the force.

Ans. $P = 247.4$ lbs., $T = 346.6$ lbs.

108. An elevator weighing 1,000 lbs. moves down with constant acceleration. It goes down by 144 feet in 12 seconds from starting. Compute the tension in the cable carrying the cage. *Ans.* $T = 937.9$ lbs.

109. An express-lift stops directly at every tenth-floor of a building. For the rise of each portion the change of velocity is such that the velocity is accelerated uniformly for the first 2.5 seconds, then it is kept constant for 10 seconds and next it is retarded with uniform rate to bring the lift to rest in the last 2.5 seconds. If the maximum velocity attained is 15 feet per second and the weight of the elevator be 1,500 lbs., calculate the tensions in the cable during the three stages of motion of the elevator.

Ans. 1779.5, 1500, 1220.5 all in pounds.

110. A mass of 15 pounds is rotated with a cord 9 ft. long in a horizontal circle with a velocity of 6 ft. per second. Find the tension in the string.

Ans. 1.86 lbs. wt.

111. To what inclination should a circular motor-track of 5 laps to a mile be banked for driving up at a speed of 30 miles per hour?

Ans. 19° — 48'.

112. What should be the least radius of a circular tram line so that a car may run on a level track at 15 miles per hour without producing a side thrust of more than one-hundredth part of its own weight?

Ans. 1503 feet.

113. On a metre-gauge line a train is running at a speed of 30 miles per hour in a circle of 350 metres radius. Find in inches, how much must the outer rail be higher than the inner rail so that there is no thrust on the flanges of the wheels. *Ans.* 2.06 inch.

114. A roadway is intercepted by a canal over which there is a bridge in the shape of a circular arc of radius 60 feet. Find the maximum speed in

miles per hour with which a motor cyclist can cross the bridge so that the cycle does not leave the bridge surface when it reaches the topmost point.

Ans. 30 miles per hour.

115. A ball gets its start from the topmost point of the smooth surface of a cylindrical body and slides down the surface—the plane of the motion is at right angles to the axis of the cylinder. The motion is accelerated due to the action of the gravity alone. Find at what position the ball will leave the cylindrical surface.

Ans. $\theta = \cos^{-1}\frac{2}{3}$ (θ is the angle made by the radial line joining the position and the axis with the vertical).

116. If the speed of a conical pendulum changes from 60 revolutions per minute to 75 revolutions per minute, what change will occur in the height?

Ans. .0013 inch.

117. In a conical pendulum the weight of the bob is 2.5 lbs. The length of the string is 1 ft. 8 in. and the radius of the horizontal circle described by the bob is 8 inches. Determine the tension in the string and the angular and linear velocities of the bob.

Ans. 2.37 lbs. 4.6°/sec.
3.06 ft./sec.

118. A metallic ball weighing 5 ounces is being rotated in a vertical circle with the help of a string 5 feet long. If the velocity at the lowest point be 40 feet per second, find its velocity and tension in the string when it is 2.5 feet and 7.5 feet above the lowest point.

Ans. 38.07 ft., per sec., 33.42 ft., per sec.
2.96 lbs. wt., 1.7 lbs. wt.

119. The breaking strength of a thread is 6 lbs. If a stone weighing 3 lbs. be tied to one end of the thread, 3 feet in length, and rotated in a vertical plane by holding at the other end, find the angular velocity at which the thread will break. *Ans.* $\omega = 3.27$ rad. per second.

120. If the stone in the previous problem rotates in a horizontal plane, find the angular velocity. *Ans.* $\omega = 4.63$ rad. per second.

121. How many complete oscillations will a pendulum 10 feet in length make in 24 hours? $g = 32.2$ per sec. per sec.

Ans. 2469.

122. Find the time for an oscillation of a pendulum 88.29 metres long in London. *Ans.* 18.84 seconds.

123. Compare the accelerations due to gravity of two places where the lengths of the seconds pendulum are 3.265 ft. and 3.257 ft. respectively.

Ans. 32.22 : 32.14.

124. Under what load at the middle of the span of a beam will the period of vibration be .3 second if the beam deflects 1 in. for a load of 1 ton?

Ans. 1973 lbs.

125. A reciprocating part of a machine weighing 250 lbs. moves approximately in a simple harmonic motion. The length of the path of travel from one to the other extreme end is 15 in. What are the magnitudes of the force acting on the part at distances of 5 and 12 in. respectively from one end, and what are the velocities at those two points if the maximum velocity developed by the part be 20.9 feet per second?

Ans. 3617 lbs., 8680 lbs.
15.59 ft. per sec., 19.15 ft. per sec.

126. A helical spring held in a vertical position is set in vibration by the application of a load of 4 lbs. If the stiffness of the spring is such that one pound of load produces a stretch of .05 foot, find the time of a complete oscillation. Neglect the consideration of the mass of the spring.

Ans. .495 second.

127. A vertical helical spring being loaded with a weight of 9 lbs. oscillates 90 times per minute. Determine the weight which if gradually applied produces an elongation of the spring by 2 inches.

Ans. 4.136 lbs.

128. In a minute 120 complete oscillations are made when a load of 15 lbs. is suspended by a spiral spring. If a load of 6.128 lbs. is gradually applied on the spring, find the stretch produced by the load.

Ans. 1 inch.

129. A weight of 5 lbs. is suspended by an elastic cord of 2 feet length and reaches the state of equilibrium after stretching the cord by 5 inches. If the load-end of the cord is pulled to stretch it further by 4 inches and the pulling force is abruptly cut off, find the time for a complete oscillation.

Ans. .357 sec.

130. The time of a complete oscillation of the liquid in a U-tube of uniform bore is 1.356 seconds. Determine what length of the U-tube is filled with the liquid.

Ans. 3 feet.

131. In a steam engine the piston with its rod weighs 300 lbs. The crank length of the engine is 9 in. The crank shaft makes 300 r.p.m. Assuming the ratio between the crank and the connecting rod as 1: infinity, calculate the effort on the piston, (1) at the beginning of the stroke, and (2) at a distance of 3 inches from the starting of the stroke.

Ans. 9234 lbs., 4596 lbs.

132. If in the above engine the ratio between the crank and the connecting rod is 1 . 5, find the efforts and the percentage change in the values

Ans The same, 4486 lbs, 2 4%

133. In a compressor the mechanism is composed of crank, connecting rod and piston arrangement The linkage is such that the ratio between the crank and the connecting rod may be taken as infinity The crank rotates at a speed of 180 r p m The crank length is 9 inches The piston with its other parts weighs 161 lbs The linkage allows to neglect the weight of the connecting rod Compute the force of the crank pin exerted on the piston when the crank-pin is 4 inches from the mid position towards the inner dead-centre, *i e*, in the direction of motion The diameter of the piston is 10 inches and the air pressure is 60 lbs per sq inch at that instant

Ans 4650 8 lbs

134 A tram car weighs 25,000 lbs of which the wheels and the truck weigh 2,500 lbs The remaining portion oscillates harmonically in the vertical direction on the springs If the amplitude of motion is 1 inch and the frequency is 2 5 cycles per second, compute the pressure on the rails

Ans 39,360 lbs to 10,640 lbs

135 A perfectly flexible cord of uniform cross section and of length l is placed on a perfectly smooth plane surface of a table top with a portion of its length a' hanging from the round edge of the table Find the velocity of the end of the cord on the table while it leaves the table edge during sliding Take the weight per unit length of the cord as w

Hint —It is a problem on uniformly varying acceleration in a straight direction No frictional resistance acts against the motion Average acceleration is the equivalent acceleration with which the cord may be taken to move

$$Ans\quad v = \sqrt{\frac{(l^2 - a^2)g}{l}}$$

136 If a ball overtake another of 4 times of its own mass moving with one third of its velocity along the same path and if the value of e be $\frac{1}{2}$, compare their velocities after the impact *Ans* 3 8

137. From 100 feet high a perfectly elastic ball is dropped on a plane inclined 45° with the horizon placed at the midway Find the horizontal displacement of the ball when it reaches the ground *Ans* 100 feet

138. An elastic ball impinges obliquely on another ball at rest of double mass in a direction making an angle of 30° with the line of impact Find in what direction the smaller ball will move after the impact $e = \frac{1}{2}$

Ans 15° deviated back from the previous direction

139. Two perfectly elastic balls of equal mass approaching each other with equal velocities collide If the directions of the velocities be 30° and 60°

respectively with the line of impact, find in what directions the balls will run after the impact and what are the changes in the velocities.

Ans. run in the parallel directions making 45° with the common normal.
1st ball's velocity changes to $.707u$ and 2nd ball's to $1.22u$.

140. There is a direct collision between two balls of equal mass moving with equal velocities on a horizontal plane—one moving eastward while the other in the north-south direction. If the coefficient of restitution be $\frac{1}{2}$, by how far is the direction of motion of the second ball turned through?

Ans. nearly 37° eastward.

141. A ball weighing 2 lbs. runs with a velocity of 40 ft. per second and overtakes another ball weighing $\frac{1}{2}$ lb., which is moving in the same direction with a velocity of 20 ft. per second. If the coefficient of restitution be $\frac{3}{4}$, find the impulse up to the moment of greatest compression and also the total impulse. What is the velocity of the smaller ball after the impact?

Ans. $\frac{8}{g}$ gravitational unit; $\frac{14}{g}$ gravitational unit & 48 ft. per second.

CHAPTER VI

WORK, POWER AND ENERGY

121. Work. If a force acts on a mass to create a motion in it, the force is said to do work. The amount of work depends on the quantity of force and the displacement for which it acts, and is directly proportional to both of them. By the proper choice of units, work can be measured by the product of the force and the displacement. Work is generally represented by Q.

122. Work — Linear Displacement.

1. When both the force and the displacement are in the same direction, the work is measured by the product of them. As the work varies directly with the force and the displacement, *i.e.*, $Q \propto P.s$, $Q = k.P.s$. Now, if the units are properly chosen, then the constant k becomes unity and $Q = P.s$, which means that if unit force acts on a mass to produce a displacement equal to unity, then one unit of work is done.

2. If the displacement and the force are not in the same direction, *i.e.*, if the force is an oblique one as is shown in Figure 62, then either the component of the force along the displacement or the component of the displacement in the direction of the force must be taken into account in measuring the work, *i.e.*, in measuring work both the displacement and the force must be calculated in the same direction. If P (Fig. 62) be the force in a direction, shown in the figure, and if s be the displacement along the plane, the work done is equal to $P \cos \theta \times s$ or, $P \times s \cos \theta$. $P \cos \theta$ is the effective force for the displacement s. $P \sin \theta$, which is at right angles to the direction of the displacement can have no effect on the motion in the direction of the displacement.

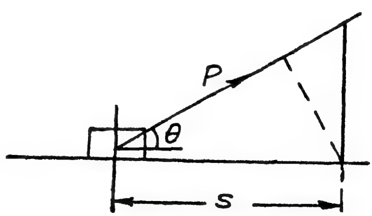


FIG. 62

123. The general form of the equation for the work done is,

$$Q = \int P.\,ds \qquad \cdots\cdots\cdots Eq.\ 61$$

(*a*) When *P* is constant,

$$Q = \int_0^s P.\,ds = P \int_0^s ds = P.\,s \qquad \cdots\cdots\cdots Eq.\ 62$$

(*b*) When *P* is constant but an oblique one,

$$Q = P \cos\theta \int_0^s ds = P\cos\theta.\,s \qquad \cdots\cdots\cdots Eq.\ 63$$

which can also be written as,

$Q = P.\,s\cos\theta$, θ being the angle of obliquity of the force *P*.

124. The Unit. In the British absolute system the unit is a foot-poundal. This is the quantity of work done by a force of one poundal to move a mass through one foot.

The engineers always choose a bigger unit, foot-pound, to measure work. A foot-pound is a quantity of work done by a force of one pound in moving a mass through one foot. One foot-pound is, therefore, equal to 32.2 foot-poundals. An inch-pound, a foot-ton, etc., are also used as units if required.

In C.G.S. system the unit of work is an erg or a centimetre-dyne. This is the work done by a force of one dyne to move a mass through one centimetre. Centimetre-gram unit is also sometimes taken as unit of work, and this is equal to 981 ergs.

Electrical engineers' unit of work is a joule, which is equivalent to 10^7 ergs. This is the smallest electrical unit.

The bigger units are Watt-hour and Kilowatt-hour which are equal to 36×10^9 and 36×10^{12} ergs respectively.

Mechanical engineers' unit of work in metric system is a Kilogram metre, which is equal to $(1000 \times 100 \times 981)$ ergs, *i.e.*, 9.81×10^7 ergs or 9.81 joules.

Illus. Ex. 64. *Find the work done on a mass of 100 pounds when lifted through 150 feet, and represent the result in various units.*

Work done $= 100 \times 150 = 15000$ ft. lbs.

$= 15000 \times 32.2 = 483000$ ft. poundals.

Again, 1 lb. $= 453.6$ grams and 1 ft. $= 30.48$ cm.

$\therefore$ Work $= 100 \times 453.6 \times 150 \times 30.48 = 143 \times 10^6$ cm. gms.

$= 143 \times 10^6 \times 981 = 138 \times 10^9$ ergs or cm. dynes.

$= (138 \times 10^9) \div 10^7 = 138 \times 10^2$ joules.

$= (138 \times 10^9) \div (36 \times 10^9) = 3.82$ watt-hour.

$= (138 \times 10^9) \div (36 \times 10^{12}) = .00382$ kilowatt-hour.

$= (138 \times 10^9) \div (9.81 \times 10^7) = 1406.7$ kg. metre.

Illus. Ex. 65. *A boy, pulls his toy-cart with a force of 5 lbs. and draws it through a distance of 50 feet. What is the work done if the drawing cord makes an angle of 30° with the horizon?*

Work $= 5 \cos 30 \times 50 = 5 \times .866 \times 50 = 216.5$ ft. lbs.

or, $\quad = 5 \times 50 \cos 30 = 5 \times 50 \times .866 = 216.5$ ft. lbs.

Illus. Ex. 66. *Find the work done to raise a body weighing 50 lbs. up an incline 50 feet high. Assume that there is no other resistance against the motion.*

Force along the plane down due to the weight, *i.e.*, the force required to drag the body up the plane $= 50 \sin \alpha$, where α is the angle of inclination of the plane.

The displacement along the plane for 50 feet height $= \dfrac{50}{\sin \alpha}$

$\therefore$ the work done $= 50 \sin \alpha \times \dfrac{50}{\sin \alpha} = 50 \times 50 = 2500$ ft. lbs.

or,

In the direction of the upward vertical displacement of 50 feet the force that is required to raise the body in the same direction is its own weight. Therefore, work $= 50 \times 50 = 2500$ ft. lbs.

125. Graphical representation of Work — Work Diagram.

Case I. Constant Force. If a curve for force against displacement be drawn, the force and the displacement being represented along two

rectangular axes Y and X respectively as shown in the diagram (Fig. 63), the curve is called the force-displacement curve or the work diagram The work done by a constant force P for a displacement S, in the same direction with P, is $P \times S$ units But the vertical length OA representing P multiplied by the horizontal length OB representing S gives the area under the curve Therefore, the area under the curve is proportionate to the work, $P \times S$ units

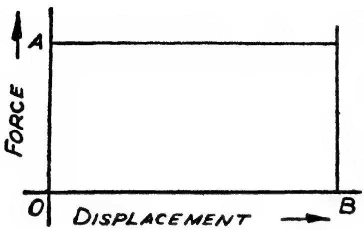


FIG 63

If the scale be chosen in such a way that 1 unit of length of the Y-axis represents p units of force and 1 unit of length of the X axis represents s units of displacement and if the area of the rectangle under the curve be A square units, then each square unit of area representing ps units of work, the whole area A square units will represent $A\ p\ s$ units of work, *i e*, $P \times S = A\ p\ s$ units of work

The amount of work represented by one square unit of the diagram is called the *scale of the diagram* Area multiplied by the scale of the diagram gives the amount of work done In the above case $p\ s$ is the scale of the diagram

A simple case of work done by a constant force is that of a mass being lifted against the gravity If the weight of the mass be w and the height through which it is lifted be h, then the work done is $w\ h$ units (gravitational)

Case II. Varying force (rate of variation is uniform)

If the force does not remain constant throughout the displacement but varies uniformly, the force-displacement curve will be something like that as is shown in Fig 64, an inclined straight line Let P_1 and P_2 be the initial and final values of the force As the force varies uniformly, the value of the force at a distance x from the origin, $P = P_1 + (P_2 - P_1) \dfrac{x}{s}$

$$\text{Therefore, } Q = \int_0^s \left[P_1 + (P_2 - P_1) . \frac{x}{s}\right] dx$$

$$= \tfrac{1}{2}(P_1 + P_2)s \qquad Eq.\ 64$$

The problem may be treated by the method of summation as follows :

Divide OB, which represents the displacement s, into n number of very small equal parts. Then each of the segments is equal to $\frac{S}{n}$.

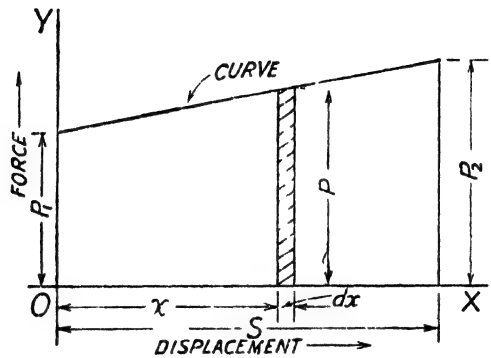


FIG 64

The total work done is equal to the sum of the works done during these small displacements Now, if P_1 be the initial force and p be the small increment in force during each of these successive displacements of $\frac{S}{n}$ and if P_2 be the final force, then $(P_2 - P_1) = np$ The initial values for the forces during these successive displacements are respectively equal to P_1, $P_1 + p$, $P_1 + 2p$, $P_1 + 3p$, ... $P_1 + (n-1)p$, and they may be taken as constant throughout the respective displacements if n be a very big number

$$\therefore \text{ the total work done} = P_1 \times \frac{S}{n} + (P_1 + p).\frac{S}{n} +$$

$$(P_1 + 2p).\frac{S}{n} + \ldots\ldots \quad \ldots [P_1 + (n-1)p].\frac{S}{n}$$

$$= n.\frac{S}{n}\cdot P_1 + \frac{S}{n}.p\,(1 + 2 + 3 + 4 + \ldots + n - 1)$$

$$= P_1S + \frac{S}{n} \cdot p \cdot \frac{n(n-1)}{2}$$

$$= P_1S + \frac{S}{2} \; p \cdot (n-1)$$

Where n is too great, n and $(n-1)$ are approximately equal.

$\therefore \; (n-1)\,p = np = (P_2 - P_1)$

$$\therefore \text{ Total work} = P_1S + \frac{S}{2}\,(P_2 - P_1)$$

$$= \frac{P_1 + P_2}{2} \cdot S$$

But $\left(\frac{P_1 + P_2}{2}\right)$ is the average force during the displacement

Therefore, the work done is equal to the average force multiplied by the displacement Again, this average force can be represented by the average height of the diagram (Fig 64) The average height multiplied by the base representing the displacement gives the area under the force-displacement curve Thus, in this case too it is found that the area under the curve represents the work done This area multiplied by the scale of the diagram gives the work done

Case III Varying force (rate of variation is not uniform)

Suppose the force-displacement curve is as shown in Fig 65, where the change of force is not uniform Divide OB, which represents S,

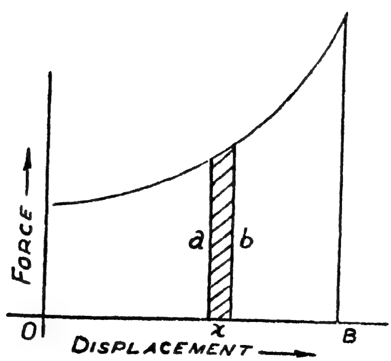


FIG. 65

into n number of small equal parts, each of value x. Let the initial force during any one of these small divisions be P_1 and the final force be P_2, which are represented by the vertical lengths a and b respectively. As x is too small the portion of the curve between the vertical lines a and b can be considered approximately as a straight line. Then the work done for this displacement is $\left(\frac{P_1 + P_2}{2}\right) \times x$, *i.e.*, $\frac{a+b}{2} . x$, which is the area under the curve of that portion. In this way, the total amount of work can be obtained by finding out the areas separately for these n number of x displacements and adding them all together and then multiplying the total area by the scale of the diagram, as was done in previous cases. Thus, the form of the equation is,

$$Q = \Sigma P . x \qquad \cdots\cdots\cdots\cdots Eq. \ 65$$

126. Indicator Diagram. In cases of heat-engines, work diagrams for the cylinders are obtained with the help of an instrument called 'Indicator', and hence the work diagrams are named as Indicator Diagrams. For one revolution of the crank, the work done within a cylinder of a steam-engine presents the work diagram similar to as shown in Fig. 66. The vertical axis represents the steam pressure and the horizontal axis represents the piston travel.

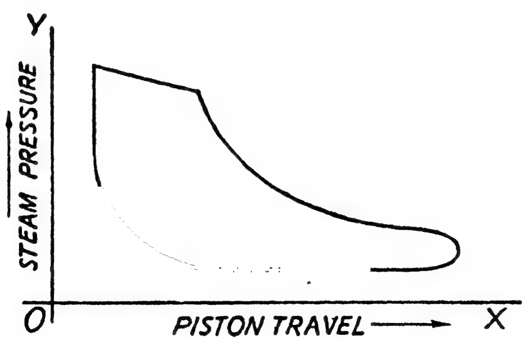


FIG. 66

If the linear scale of the diagram be 1″ $= x$ inches, and the force scale be 1″ $= y$ lbs., then 1 sq. inch of the diagram will represent xy inch-pounds, which is the scale of the diagram. Now, if the area of the diagram be A sq. inches, which can be measured by a plani-

meter and if the stroke be L inches, then the average height of the diagram will be, $(A \div \frac{L}{x})$ inches. Therefore, the average or the mean pressure of the fluid is $(A \div \frac{L}{x}).y$ lbs. per sq. inch. Let it be represented by p. If a be the area of the piston in sq. inches, the average force exerted on the piston throughout the stroke is equal to $p.a$ lbs. and the work done for one stroke is $p\,L\,a$ inch pounds. If in a revolution of the crank only one of the two strokes be effective and if the crank rotates n times per minute, then the work done per minute is equal to $p\,L\,a\,n$ inch pounds or $p\,l\,a\,n$ foot pounds, where L inches $= l$ ft. The equation may be put as,

$$Q = p\,l\,a\,n \qquad \text{Eq. 66}$$

If both the strokes be effective and if the pressure surfaces on the piston for both the strokes be the same, then the work done per minute is equal to $2p\,l\,a\,n$ foot pounds, *i.e.*,

$$Q = 2\,p\,l\,a\,n \qquad \text{Eq. 67}$$

Note that, p = fluid pressure per sq. inch in lbs.

l = length of the stroke in feet

a = area of the piston in sq. inches

n = number of rotations per minute.

WORK—ANGULAR DISPLACEMENT

127. Moment of a Force. If a force be applied on a particle to create the tendency of rotation about a fixed point or axis the force is said to produce a moment. Moment is the measure of the turning tendency of the particle about the fixed point or the axis. When the moment causes rotation it is said to do work. Moment is measured by the product of the force and the perpendicular distance of its line of action from the fixed point or the axis. It is called twisting moment, turning moment or torque, and is generally represented by the letter T. In the British absolute system the unit is a poundal-foot; it is the measure when a force of one poundal acts at a perpendicular distance of one foot. In gravitational system it is one pound-foot. Smaller unit, pound-inch, is also used to

measure a torque when required. In C. G. S. system the unit is a dyne-centimetre.

If O be the definite point (Fig. 67) about which a particle rotates due to the action of a force P and if R be the perpendicular distance of the line of action of the force P from the point O, then the turning moment of P about O is $P \times R$. The direction is clockwise or anti-clockwise according as O lies to the right or left of the line of action of the force. In adding moments the clockwise and the anti-clockwise moments must be considered as of opposite signs. Whether the clockwise moments will be taken as positive or the anti-clockwise moments positive, is immaterial. Generally the clockwise moments are reckoned as positive.

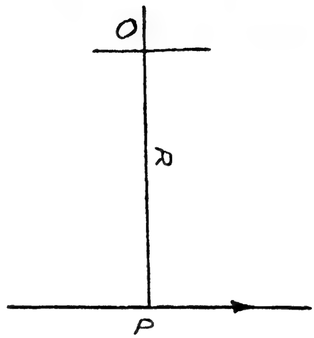


Fig. 67

In case where the particle rotates about a fixed axis the line of action of the force may be considered in a plane normal to the axis. Let the plane cut the axis at O (Fig. 68). Then, the moment of the

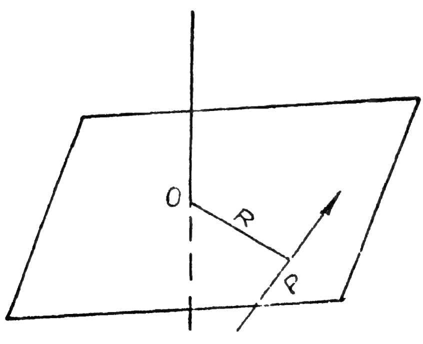


Fig. 68

force about the point O is the moment about the axis in which O is a point, and it is equal to $P \times R$, where R is the perpendicular distance between the line of action of the force P and the axis.

128. Work done on a Particle rotating in a Circle. If a particle is constrained to rotate in a definite circular path, the effective force to move the particle along the path must be always tangential to the path of rotation. Whether the acting force is constant or variable, the magnitude of the force can be taken as constant for a small angular displacement $d\,\theta$ radian. Therefore, the work done for this differential angular displacement $d\,\theta$ is $P \times d\,\theta . r$, where P is the effective force to produce the displacement $d\,\theta$ and r is the radius of the circular path. Hence, the work done for an angular displacement may be put in the form of the equation,

$$Q = \int P . d\,\theta . r = \int T . d\,\theta \qquad Eq.\ 68$$

where T represents the moment of the force or torque about the centre or axis of rotation.

(*a*) When T is constant and is represented by T_c, θ is the angular displacement in radians corresponding to the linear displacement P_1P_2 (Fig. 41),

$$Q = T_c \int_0^{\theta} d\,\theta = T_c . \theta \qquad \cdots \qquad Eq.\ 69$$

(*b*) When T varies uniformly from T_i to T_F, the initial to the final torque respectively,

$$Q = \tfrac{1}{2}\,(T_i + T_F)\,\theta \qquad \cdots \qquad Eq.\ 70$$

(*c*) In case where T_i in the above equation is zero,

$$Q = \tfrac{1}{2}\,T_F . \theta \qquad \cdots\cdots\cdots \qquad Eq.\ 71$$

T_F represents the final torque changing uniformly from zero.

If the angular displacement is given by the number of rotations, N, a particle undergoes, then the angular displacement being equal to $2\,\pi\,N$ radians, the equations 69, 70 and 71 respectively become,

$$Q = 2\,\pi\,T_c\,N \qquad \cdots\cdots\cdots Eq.\ 72$$

$$Q = \pi\,(T_i + T_F)\,N \qquad \cdots\cdots\cdots Eq.\ 73$$

and $$Q = \pi\,T_F\,N \qquad \cdots\cdots\cdots Eq.\ 74$$

129. In case of big masses the force may be applied in two different ways. Firstly, the force may act at the same point of the body, as in the case of a wheel being rotated with the help of a handle ; and secondly, it may act at successive points, as in the case of toothed wheels. It should be noted that in both the cases the effective force acts tangentially to the path of rotation.

According to the unit of torque the work done will be in foot-poundals, foot-pounds, inch-pound, centimetre-dyne etc.

Illus. Ex. 67. *What is the work done if the torque of a steam-engine shaft changes uniformly from 1000 to 1500 lb. ft. in 100 revolutions of the shaft?*

Work $= (1000 + 1500)\,\pi . 100 = 785000$ foot-pounds.
(*Eq.* 73)

Illus. Ex. 68. *If torque is proportional to the angle of twist, find the work done in twisting a shaft through 10 degrees. The torque at the finish is 200 lb. inches.*

Torque is proportional to the angle of twist. Therefore, the initial torque is zero.

The angle of twist is $10^\circ = \frac{\pi}{18}$ radians.

$\therefore$ the work done $= \frac{1}{2} \times 200 \times \frac{\pi}{18} = 17$ in. lbs.
(*Eq.* 71)

Illus. Ex. 69. *If a force of 100 lbs. acts on a body and causes it to rotate about an axis, and if the force always acts tangentially at a distance of one foot from the axis, what amount of work is done by the force when the body makes 100 turns?*

The work done $= 100 \times 2\,\pi \times 1 \times 100 = 62800$ ft. lbs.

Illus. Ex. 70. *If a force of 5000 dynes be required to rotate a 1-meter handle of a lifting machine and if 100 turns are required to raise a load through 6.28 m., find the load.*

Torque $= 5000 \times 100$ centimetre-dynes.

$\therefore$ the work done $= (5000 \times 100 \times 2\,\pi \times 100)$ ergs.

$\therefore$ the load lifted $= \dfrac{5000 \times 100 \times 2 \times 3.14 \times 100}{6.28 \times 100 \times 1000}$

$= 500$ kilogrammes.

POWER

130. Power. Power is the time rate of doing work. It indicates the quantity of work that can be done per unit of time.

131. The Unit. The unit of power in the British system is 550 ft. lbs. of work per second or 33000 ft. lbs. of work per minute. Watt found that a good working horse could do 33000 ft. lbs. of work per minute on the average, and he introduced this as the unit for measuring the working capacity of steam engines. From that time this unit has been universally accepted as the unit of power, and is named as Horse Power. It is generally denoted by *H.P.*, the initials.

In C. G. S. system the unit of power is one erg per second.

The electrical engineers' unit of power is a watt, which is equivalent to a rate of 10^7 ergs per second, *i.e.*, one joule per second. Engineers frequently use a bigger unit, a kilowatt, which is equal to 1000 watts.

One horse power is equivalent to 746 watts or .746 kilowatt. One kilowatt is equivalent to 1.34 *H.P.* One joule is about .737 of a foot pound. Hence, a watt is .737 ft. lb. per second or, 1 ft. lb. per second is equivalent to 1.356 watts.

The mechanical engineers' unit of power in metric system is 75 kilogramme-metre per second. This is also called one horse power. The relation between this horse power and the previous one is that the previous one is 1.0136 times the latter. This unit is approximately equivalent to 736 watts or .736 kilowatt.

In the Continent the customary unit for measuring power is the horse power equivalent to 75 kilogramme-metres per second. But in British Isles and the United States of America and the countries under their control the unit that is adopted to measure power is the horse power equivalent to 550 ft. lbs. per second.

132. Power-Diagram. Just similar to a work-diagram, a power-diagram can be drawn—the Y-axis represents the power and the X-axis the distance or the time as required. In the power-displace-

ment diagram the variation of power at different points in the path is found, and the power-time diagram indicates the variation at different instants.

Illus. Ex. 71. *A train is running with a constant speed of 60 miles per hour. If the weight of the train be 250 tons and the resistance due to air and friction be 20 lbs. per ton, what H.P. should be maintained by the engine? Convert the result in the Continental unit.*

$W = 250$ tons.

$v = 60$ miles per hr. $= 88$ ft. per sec.

$R =$ Force applied $= (20 \times 250)$ lbs.

$$H.\ P. = \frac{\text{work done per sec. in ft. lbs.}}{550}$$

$$= \frac{20 \times 250 \times 88}{550} = 800.$$

In Continental system it is equal to $800 \times \frac{746}{736} = 810.65.$

Illus. Ex. 72. *A man whose weight is 11 stones rises up a hill with a slope of 1 in 5 at the rate of 1.5 miles per hour. What fraction of a horse power is he maintaining?*

$W = 11$ stones

$v = 1.5$ miles per hr. $= 2.2$ ft. per sec.

Force applied $=$ the component of the weight down the plane.

$= W \sin \alpha$

$= 11 \times 14 \times \frac{1}{5}$ lbs.

$$H.\ P. = \frac{\text{work in ft. lbs. per sec}}{550}$$

$$= \frac{11 \times 14 \times 2.2}{5 \times 550} = .123.$$

Illus. Ex. 73. *What should be the theoretical H.P. of a motor in a pumping set which can fill a tank, 10 feet cube, on the top of a tower 50 ft. high, in 1 hr. 40 mins., if the set be fitted with a tube-well where the surface of water always remains constant at 15 ft. below the ground level? Express the result in electrical engineers' unit. (one cu. ft. of water weighs 62.5 lbs.)*

Volume of water raised $=$ the capacity of the tank $= 10^3$ cu. ft.

$\therefore$ the weight of water raised $= 10^3 \times 62.5$ lbs.

Height through which water is lifted $= (50 + 15 + 10)$.

$= 75$ feet.

$\therefore$ the total work done $= (10^3 \times 62.5 \times 75)$ ft. lbs.

$$\therefore \text{ the quantity of work done per minute} = \frac{(10^3 \times 62.5 \times 75)}{100}$$

$$\therefore \; H.P. \text{ required} = \frac{10^3 \times 62.5 \times 75}{100 \times 33000} = 1.4204.$$

Now, 1.4204 *H.P.* $= 1.4204 \times 746 = 1059.6$ watts, *i.e.*, 1.4204 *H.P.* or 1.0596 k.w. motor is required for the purpose.

Illus. Ex. 74. *Find the horse power absorbed by a lathe on work if the tensions in the two ends of the belt be 150 and 50 lbs. respectively and if the diameter of the driven wheel be 10 ins. The pulley makes 100 revolutions per minute* (Fig. 69).

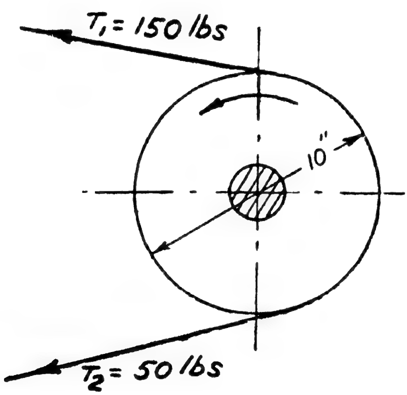


FIG. 69

Torques due to forces are respectively equal to (150×5) and (50×5) lb. inches.

$\therefore$ the work done per minute $= 2\pi\,(150 \times 5 - 50 \times 5) \times 100$ in. lbs.

$$\therefore \; H.P. = \frac{2\pi\,(150 \times 5 - 50 \times 5) \times 100}{12 \times 33000} = .79$$

Illus. Ex. 75. *Find the amount of force in action on the surface of a line-shaft of a workshop when it is driving different machines requiring a total power of 30 H. P. The number of revolutions of the shaft is observed to be 100 per minute. The diameter of the shaft is 2.5 inches.*

$H.\ P. = 30$

$D = 2.5$ ins. $\therefore$ Radius $= 1.25$ ins.

$$H.\ P. = \frac{2\pi\ T N}{12 \times 33000}$$

$N = 100$

$$\therefore\ T = \frac{H\ P \times 12 \times 33000}{2\pi\ N}$$

But $T = P \times$ radius

$$\therefore\ P = \frac{H.P. \times 12 \times 33000}{2\pi \times N \times \text{radius}}$$

Now, substituting the values of $H.P$, N and radius

$$P = \frac{30 \times 12 \times 33000}{2 \times 3.14 \times 100 \times 1.25}$$

$= 7570$ lbs.

133. Energy. When a particle has got the capacity for doing work, the particle is said to possess *energy*. Energy of a particle determines the total quantity of work it can do. The capacity for doing work may be due to various reasons, such as, position, motion, temperature, chemical composition, having higher electric potential, etc. For instance, a body raised through a height, a body in motion, a compressed helical spring, compressed air, coal, storage battery—all possess energy. In this book, however, only two kinds of mechanical energy—potential and kinetic—will be discussed.

134. The Unit. Because energy of a particle is nothing but the total amount of *work* it can do, it is measured in the same units with work.

135. Potential Energy. A particle is said to possess potential energy when it can do work due to its relative position with respect to the zero level. Sea level is taken as the zero level. When a particle weighing w is raised from the zero level to a height h, the quantity of work done is equal to wh (gravitational unit), and it is evident that in falling through the same height the particle can do the same

quantity of work—the amount of force and the distance travelled in both the cases are the same. Due to its raised position it has got the capacity for doing work which varies directly with the height. This capacity for doing work due to position only is termed as *Potential Energy*.

136. Kinetic Energy—straight motion.

Capacity of a particle for doing work by virtue of its motion is known as *Kinetic Energy.*

If a force P acts upon a free particle of mass m,

$$P = m. \frac{dv}{dt}$$

$$\text{or,} \quad P.\ dt = m.\ dv$$

$$\text{or,} \quad P.\ v.\ dt = m.\ v.\ dv$$

$$\text{But} \quad v.\ dt = ds$$

$$\text{Therefore,} \quad P.\ ds = m.\ v.\ dv$$

$$\text{or,} \quad \int_0^s P.\ ds = \int_0^v m.\ v.\ dv \qquad \ldots\ldots (i)$$

$$\text{or,} \quad P.\ s = \tfrac{1}{2}\, m.\ v^2$$

That is, $P.\ s$ is the quantity of work done on the particle to create a velocity v. It is evident that to bring the particle in motion to rest from the velocity v, the particle can do the same quantity of work against a resistance. Hence the kinetic energy,

$$K.E. = \tfrac{1}{2}\, m.\, v^2 \qquad \ldots\ldots\ldots\ldots Eq.\ 75$$

If the lower limits in the form (i) be s_1 & v_1 and the upper limits be s_2 & v_2 respectively,

$$\int_{s_1}^{s_2} P.\ ds \qquad \int_{v_1}^{v_2} m.\ v.\ dv$$

$$\text{or,} \quad P\ (s_2 - s_1) = \tfrac{1}{2}\, m\ (v_2^2 - v_1^2) \qquad \ldots\ldots\ldots\ldots Eq.\ 76$$

$$= \tfrac{1}{2}\, m\ v_2^2 - \tfrac{1}{2}\, m\ v_1^2$$

$$= K.E._F - K.E._i,$$

where $K.E._F$ and $K.E._i$ are the final and initial kinetic energies respectively, and this is the change in the kinetic energy of the particle.

Eq. 76 means that the force P acting for a displacement $(s_2 - s_1)$ changes the velocity of the particle from v_1 to v_2.

The problem can be treated also as follows :—

Suppose a force P_1 acts on a particle whose mass is m for time t_1 changing the velocity of the mass from zero to v, when the accelerating force is withdrawn and a resistance, *i.e.*, a retarding force P_2 is applied on the particle for time t_2 to bring it to rest again. Then, $P_1 t_1 = m v = P_2 t_2$.

Now, the velocity changes from o to v and from v to o in times t_1 and t_2 respectively. Therefore, the average velocity during these periods is $\frac{1}{2} v$, and hence the spaces described during these times are respectively equal to $\frac{1}{2} v . t_1$ and $\frac{1}{2} v . t_2$. Thus, the work done by the force P_1 on the particle is equal to

$$P_1 \times \frac{1}{2} v t_1 \times \frac{1}{2} v - \frac{1}{2} m v^2$$

$$(P_1 t_1 \text{ being equal to } m v).$$

Again, when the particle attains this velocity v, the force P_1 is withdrawn and it moves against a resistance P_2 for time t_2 before coming to rest. Therefore, the work done by the particle against the resistance due to its motion is equal to

$$P_2 \times \frac{1}{2} v t_2 = P_2 t_2 \times \frac{1}{2} v = \frac{1}{2} m v^2$$

$$(P_2 t_2 \text{ being equal to } m v).$$

Thus, the quantity of work done on the particle creates this velocity and due to this motion the particle is capable of doing exactly an equal amount of work before coming to rest. This capacity for doing work due to motion only is called the kinetic energy, and the symbolic representation of the quantity is $\frac{1}{2} m v^2$. Kinetic energy is denoted by the initials $K.E.$

If due to the action of a force the velocity of a body changes from v_1 to v_2, the change in the kinetic energy will, then, be equal to $\frac{1}{2} m (v_2^2 - v_1^2)$. This change is either a gain or a loss according

te as v_2 is greater or smaller than v_1. In case of a loss the result will be a negative one.

137. Energy of a falling body at a definite instant. If a particle *of weight W rests at a height h from the zero level, the energy* possessed by the particle is fully of potential nature and is equal to Wh (Fig. 70). Now if the particle falls freely through a distance x, its velocity is changed from o to $\sqrt{2gx}$. At this position the particle possesses two kinds of energy—one due to its position at a height $(h - x)$, and the other due to its motion at this height. The potential energy at this position is equal to $W(h - x)$ and the kinetic energy is equal to $\frac{1}{2} \cdot \frac{W}{g} \cdot (\sqrt{2gx})^2 = Wx$.

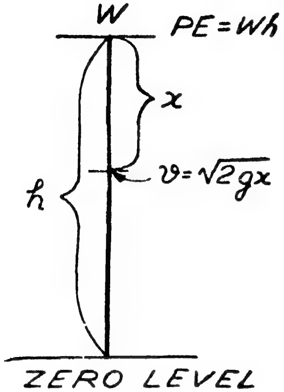


FIG. 70

Hence, the total quantity of energy possessed by the particle at this position is equal to $W(h - x) + Wx = Wh$, which is the same as possessed by the particle while at rest at the height h. However, it is found that there is no change in the total quantity and it remains constant for any position of the particle, because for any value of x the total quantity of energy is equal to Wh.

138. Conservation of Energy. In cases where the bodies in motion are brought to rest by a resisting force of frictional kind, such as the force applied by means of brakes in stopping a vehicle, it is found apparently that there is a loss in the mechanical energy. Apparently though there is a loss, heat is generally found to be generated at the rubbing surface and on experiment it is found that this heat is equivalent to the quantity of the mechanical energy that disappears. The experiments of Joule and other scientists have proved that the visible kinetic energy of the moving bodies is changed to invisible energy of the molecules of those bodies.

Thus, in all cases the energy merely changes its form. Maxwell states, "The total quantity of energy in a system of bodies can neither be increased nor diminished by any action between the parts of the

system though it may be transferred into another form of which energy is susceptible." This is what is known as the *Conservation of Energy*.

139. Graphical representation of the quantity of Work done by a moving particle against resistance. Fig. 71 represents a case where force is acting on a particle to move it against resistance. Y-axis represents force and resistance, and the X-axis represents the

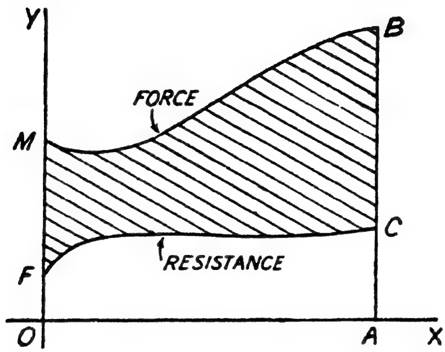


FIG. 71

displacement. The total work done on the body for a displacement OA is represented by the area $ABMO$, but of this, the quantity of work represented by the area $ACFO$ goes to overcome the resistance. Therefore, the net work done or the kinetic energy stored up in the body is represented by the shaded area $CBMF$.

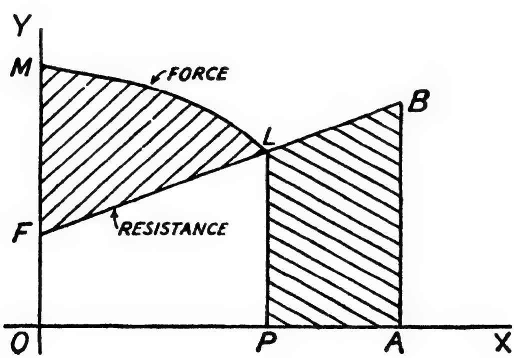


FIG. 72

Fig. 72 represents a case where a varying force is applied on a body to move it against uniformly varying resistance and after a period the force is withdrawn and the body comes to rest due to the action of the resistive force. For the displacement *OP*, the *K.E.* of the body is represented by the shaded area *MLF*. But after this displacement the action of the force is withdrawn and the resistance (which is a variable one as is shown in the diagram—in this case it is increasing with the displacement) creates retardation to bring the body to rest. Thus, the shaded area *BLPA* during the displacement *PA*, representing the work done by the body against the resistance due to its motion, must be equal to the shaded area *MLF*, which represents the kinetic energy of the body stored up during the displacement *OP*. The total displacement, starting from rest and coming to rest again is $OP + PA = OA$.

Illus. Ex. 76. *A motor car attains a speed of 30 miles per hour in 60 seconds, during which it covers 1000 yards. If the weight of the car be 5 tons and the space-average resistance be 20 lbs. per ton, find the average horse-power of the engine.*

$$K.\ E. \text{ of the car at 30 miles speed} = \tfrac{1}{2} \times \frac{5 \times 2240}{32.2} \times (44)^2 \text{ ft. lbs.}$$

If P_f be the average accelerating force, the work done on the car to change the velocity from zero to 30 miles per hour in addition to the work done against the resistance is equal to $P_f \times 1000 \times 3$ ft lbs., and this work is stored as *K. E.* in the car.

$$\therefore\ P_f \times 1000 \times 3 = \tfrac{1}{2} \times \frac{5 \times 2240 \times 44 \times 44}{32.2}$$

From which $P_f = 112.3$ lbs.

If P_T be the total force applied, then $P_T = P_f + R_T$

where R_T is the total resistance.

$$\therefore\ P_T = 112.3 + (20 \times 5) = 212.3 \text{ lbs.}$$

Then the total work done by the engine in 60 seconds is equal to $212.3 \times 1000 \times 3$ ft. lbs. Therefore, H. P. of the engine is equal to $(212.3 \times 1000 \times 3) \div (60 \times 550) = 19.3$.

Illus. Ex. 77. *A train whose weight is 200 tons, after running at a constant speed of 60 miles per hour on a level track rises up a slope 1 in 500 for 8 minutes and 20 seconds, where it again gets a level track. If the force applied by the engine and the track resistance remain constant, find the drop in the kinetic energy to go up the slope.*

When the train goes up the slope, the retarding force is equal to

$$W \sin \alpha, \textit{ i.e., } 200 \times 2240 \times \frac{1}{500}$$

Therefore, the retardation is equal to

$$\frac{200 \times 2240}{500} \div \frac{200 \times 2240}{32.2} = .0644 \text{ ft per sec. per sec.}$$

The velocity after 8 min. 20 secs.,

$$= 88 - (.0644 \times 500) = 55.8 \text{ ft. per sec.}$$

$$\therefore \text{ the change in } K.\ E. = \tfrac{1}{2} \times \frac{200 \times 2240}{32.2} \{(55.8)^2 - (88)^2\}$$

$$= -3221000 \text{ ft. lbs.}$$

i.e., the loss is 3221000 ft. lbs.

Illus. Ex. 78. *A railway engine draws a composition weighing 100 tons up an incline 100 feet high when the tie-rod breaks and the composition falls back down the plane. If the work done during the rise is 10100 ft. tons and if the resistance due to friction and other causes against the motion during the rise as well as fall remains constant, find the velocity of the composition in miles per hour when it reaches the foot of the plane. Again, if a brake is applied during the fall and the velocity at the foot be 5 miles per hour, find the work absorbed by the brake.*

The work required to raise the composition up the plane $= 100 \times 100 = 10000$ ft. tons. But the work spent to do the same task is found to be 10100 ft. tons. Therefore, $(10100 - 10000) = 100$ ft. tons are spent to overcome the resistance.

Now, in coming down the plane the composition can do 10000 ft. tons of work; but out of it, 100 ft. tons are required to overcome the resistance. Therefore, $(10000 - 100) = 9900$ ft. tons will create motion in it. If v be the velocity in ft. per sec. at the foot of the plane and because the composition starts from rest,

$$9900 \times 2240 = \tfrac{1}{2} \times \frac{100 \times 2240}{32.2} v^2$$

From which $v = 79.84$ ft. per sec. $= 79.84 \times \frac{15}{22} = 54.44$ miles per hr.

Again, due to the application of the brake the velocity at the foot is restricted to be developed to 5 miles per hour. Therefore, the quantity of work absorbed by the brake is equal to the loss in the kinetic energy and is equal to

$$\tfrac{1}{2} \times \frac{100 \times 2240}{32.2}\{(79.84)^2 - (7.33)^2\} = 21990000 \text{ ft lbs}$$

$$= 21990000 - 2240 = 9814 \text{ ft tons}$$

Remark From the study of energy it may be summarised that,

(*a*) Work is necessary to change the position of a particle, and in changing the position a particle can do work.

(*b*) Work must be done in order to increase the velocity of a particle, and a particle, in having its velocity decreased, can do work.

Illus. Ex. 79. *Two loads are suspended as shown in Fig 73. The two pulleys A & B are fixed at a certain horizontal distance and are free to rotate about their own axes. One end of the rope is fixed while at the other end a load of 500 lbs is suspended. Another load of 1200 lbs is suspended from a movable pulley C, as shown. Find the velocity of the 500 lb load after it has moved through 20 ft. Find also the tension in the rope.*

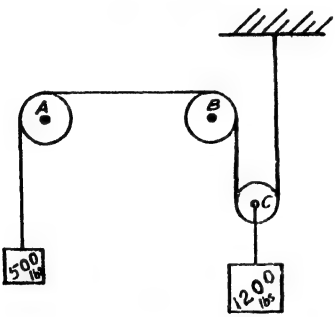


FIG 73

From the arrangement it is clear that the smaller weight will go up while the other will go down which is, therefore, the working load.

In going down the bigger load will create,

(*i*) K. E. in the smaller weight,

(*ii*) K. E. in the bigger weight

and (*iii*) will work to raise the smaller weight through 20 ft.

From the arrangement of the system it is also clear that the velocity and displacement of the bigger weight is half of those of the smaller weight.

Hence, $\tfrac{1}{2}\,\dfrac{500}{32.2}\,v^2 + \tfrac{1}{2}\,\dfrac{1200}{32.2}\left(\dfrac{v}{2}\right)^2 + 500 \times 20 = 1200 \times 10$

Solving the equation, $v^2 = 161$, *i.e.*, $v = 12.7$ ft per sec.

Again, if T be the tension in the rope,

$$T = \left(\frac{500}{32.2} \times \frac{161}{2 \times 20}\right) + 500 = 562.5 \text{ lbs}$$

KINETIC ENERGY—ANGULAR VELOCITY

140. Moment of Inertia. The property of mass of resisting any change in rotational motion is known as *moment of inertia*.

If a particle of mass m is situated at a perpendicular distance r from an axis OO' (Fig. 74) then its moment of inertia about that axis is measured by the product of the mass and the square of the perpendicular distance of the mass from the axis about which it is to rotate. The moment of inertia is, therefore, equal to mr^2 (whether the particle is in motion or not is immaterial). It is generally represented by the letter I.

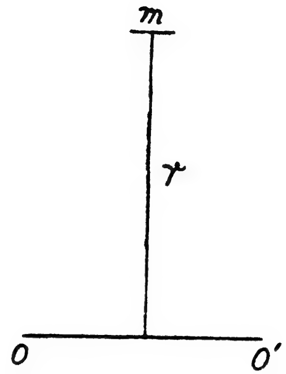


FIG. 74

141. The Unit. There is no definite name given to the unit of the moment of inertia. The unit depends on the unit of mass and the unit of length. One unit of moment of inertia is the quantity possessed by unit mass at unit distance from the axis about which it is to rotate. Gravitational or engineers' unit in F. P. S. system of the moment of inertia about an axis is the quantity possessed by unit mass (32.2 lbs.) at a distance of 1 foot from the axis. The measure is represented as one unit, two units, three units, and so on.

142. If a particle of mass m moves with a linear velocity v in a circle of radius r, its corresponding angular velocity $\omega = \frac{v}{r}$ and its kinetic energy,

$$K.E. = \tfrac{1}{2}\, m\, v^2 = \tfrac{1}{2}\, m\, \omega^2 r^2 = \tfrac{1}{2}\, I\, \omega^2, \qquad \cdots\cdots\cdots\cdots Eq.\ 77$$

where I is the moment of inertia of the mass about an axis at right angles to the plane of rotation and passing through centre (v being equal to $\omega .\ r$).

It is to be marked here that the kinetic energy at an instant due to linear velocity is $\frac{1}{2}m\ v^2$ and due to angular velocity is $\frac{1}{2}\ I\ \omega^2$. The former varies directly with the mass, whereas, the latter is directly proportional to the moment of inertia of the mass about the axis.

143. If work be done on a particle of mass m, rotating about an axis with an angular velocity of ω_1, its velocity will be changed, say,

to ω_2. If the moment of inertia of the mass be represented by I then energy of the particle will be changed by,

$$K.E._{change} = \tfrac{1}{2} I (\omega_2{}^2 - \omega_1{}^2) \qquad \text{............} Eq.\ 78$$

If a body rotating with an angular velocity ω_1, is opposed by a tangential force at the surface of rotation for a certain period, a portion of the energy is absorbed in overcoming the resistance and its velocity will be reduced to ω_2. The loss in the kinetic energy is equal to the amount of the work done against the resistance.

Illus. Ex. 80. *Find the kinetic enetgy of a fly-wheel when it is rotating at 100 revolutions per minute. The moment of inertia of the wheel is 3 gravitational units.*

$$\omega = (100 \times 2\pi) \div 60 = \frac{10\pi}{3} \text{ radians per second}$$

$$\therefore \quad K.E. = \tfrac{1}{2} \times 3 \times \left(\frac{10\pi}{3}\right)^2 = 165 \text{ foot pounds.}$$

Illus. Ex. 81. *A weight of 150 lbs. is suspended from 2.5 in. shaft of a fly-wheel in such a way that when the suspended mass falls downwards the fly-wheel rotates unwinding the cord of suspension which is wrapped round the spindle and ultimately detaches it from the shaft. If the detachment is made after 15 seconds during which the body falls through 10 feet and if the bearing resistance is 10 lbs. which acts at the surface of the shaft, find the moment of inertia of the fly-wheel.*

The average velocity of the falling mass $= \dfrac{10}{15} = \tfrac{2}{3}$ ft. per sec.

Therefore, the maximum velocity, *i.e.*, the velocity when the detachment occurs is equal to $2 \times \tfrac{2}{3} = \tfrac{4}{3}$ ft. per sec.

$\therefore$ the kinetic energy of the falling mass at the instant of detachment

$$= \tfrac{1}{2} \times \frac{150}{32.2} \times \left(\frac{4}{3}\right)^2 = 4.14 \text{ ft. lbs.}$$

Again, the net work done by the weight in falling through 10 feet by overcoming the resistance $= (150 - 10) \times 10 = 1400$ ft. lbs.

But, the net work done $=$ *K.E.* of the fly-wheel $+$ *K.E.* of the falling mass.

$$\therefore \quad 1400 = K.E._{fly\text{-}wheel} + 4.14,$$

$$\text{or, } K.E._{fly\text{-}wheel} = 1400 - 4.14 = 1395.86 \text{ ft. lbs.,}$$

and which is equal to $\tfrac{1}{2} I \omega^2$, where I is the moment of inertia of the fly-wheel and ω is the angular velocity at the moment of detachment. Then ω is equal to the linear velocity of the falling mass at that instant divided by the radius of the shaft.

$$\therefore \quad \omega = \left(\frac{4}{3}\right) \div \left(\frac{1.25}{12}\right) = 12.8 \text{ radians per second.}$$

$$I = \frac{1395.86 \times 2}{(12.8)^2} = 17 \text{ gravitational units.}$$

KINETIC ENERGY DUE TO SIMPLE HARMONIC MOTION

144. Let e be the force acting at unit distance and m be the mass moved. Because the force is directly proportional to the distance from the mid-position the amount of force acting will then be $e.x$, where x is any distance, and, therefore, at the neutral position it is zero and at the extreme end it is $e.a$, varying uniformly along the amplitude. Then the work done in moving from the neutral position to the distance x is equal to $\frac{1}{2}\ e.x.x = \frac{1}{2}\ ex^2$. This quantity of work is stored up in the mass in some form which is neither kinetic nor potential. In case of a helical spring this is called *intrinsic energy*. This energy reaches its maximum value when x is maximum and it then becomes equal to $\frac{1}{2}\ ea^2$. This quantity must be equal to the maximum amount of kinetic energy created in the mass during the motion. If v be the maximum velocity, which must be at the neutral position, the maximum kinetic energy is equal to $\frac{1}{2}\, m\, v^2$, and this is equal to $\frac{1}{2}\ ea^2$.

145. If the force-displacement curve be drawn, the vertical axis representing the force and the horizontal axis representing the displacement, the curve will be as shown in the diagram (Fig. 75).

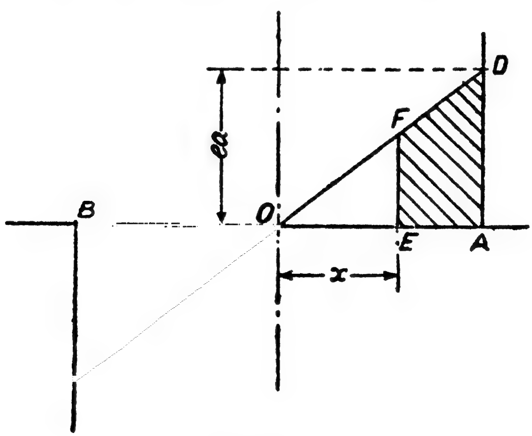


FIG. 75

The work represented by the area $\frac{1}{2}$ $(AD \times AO)$ on the right hand side represents $\frac{1}{2}\, e\, a^2$ as well as $\frac{1}{2}\, m\, v^2$. The velocity is maximum at

the point O. Therefore, the kinetic energy is maximum there. At any other point E, the work done is represented by the area $\frac{1}{2}(OE \times EF)$, but the total kinetic energy is represented by the area $\frac{1}{2}(OA \times AD)$, and therefore, the kinetic energy at the point E is represented by the area $\frac{1}{2}\{(OA \times AD) - (OE \times EF)\}$ = the area $ADFE = \frac{1}{2}e(a^2 - x^2)$.

$$\text{Again, } e = \frac{w}{g}\,\omega^2$$

$$\therefore \quad K.E. = \frac{1}{2}\,\frac{w}{g}\,\omega^2\,(a^2 - x^2)$$

where $x = 0$, $K.\ E.$ is maximum

$$\text{and } K.E. = \tfrac{1}{2}\,ea^2, \text{ i.e., } \tfrac{1}{2}\,\frac{w}{g}\,\omega^2 a^2 \qquad Eq.\ 79$$

where $x = a$, $K.E.$ is zero.

146. Helical Spring. It is easy to understand that the displacement of a point in an elastic body, such as, rubber cord, helical spring, etc., is directly proportional to its distance from the fixed end, and therefore, the velocity of a point is also directly proportional to its distance from the fixed point. The velocity of the fixed point is zero and the velocity of the free end is maximum.

If a helical spring is suspended and a mass of weight W is put at the free end to set the whole system in vibration as is shown in Fig. 76 the maximum kinetic energy of the system is $\frac{1}{2} \cdot \frac{W}{g} \cdot V^2$, where V is the maximum velocity of the free end—the mass of the spring itself is neglected, because generally the mass of the spring is very small in comparison with the mass suspended. Now, it is clear from the last article that the quantity $\frac{1}{2}\,\frac{W}{g}\,V^2$ is equal to $\frac{1}{2}\,e\,a^2$, where e is the stiffness of the spring and a is its maximum elongation. If the mass of the spring is taken into account, it is treated as follows :

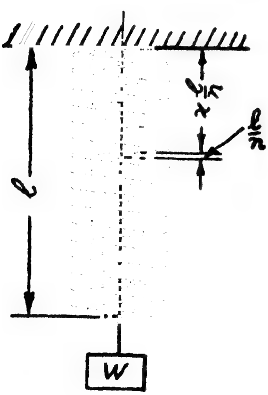


FIG. 76

Let l be the length of the spring. Take a differential length of it, dx at a distance x from the fixed point.

As velocity is proportional to the distance, the velocity of dx when W acquires the maximum velocity, V, is equal to $V \cdot \frac{x}{l}$. If w be the weight per unit of axial length of the spring, the weight of dx is equal to $w.dx$.

Therefore, the maximum kinetic energy of the whole spring is equal to $\int_0^l \frac{1}{2}\, dx \, \frac{w}{g} \left(\frac{x}{l} . V\right)^2 = \frac{1}{2}.\frac{1}{3}.\frac{w}{g}\, l\, V^2 = \frac{1}{2}.\frac{1}{3}.\frac{W_1}{g} V^2$, where W_1 is the weight of the spring.

Hence, when the mass of the spring is taken into account the total maximum kinetic energy can be expressed by the equation,

$$K.E._{max.} = \frac{1}{2g}\left(W + \frac{W_1}{3}\right). V^2 \qquad \ldots\ldots \text{Eq. } 80$$

Or, it may be proved otherwise as follows :

Let l be the length of the spring (Fig. 76). Divide it into n number of small equal parts, each of which, then, is equal to $\frac{l}{n}$

Now, the velocity being proportional to length, the velocity of the xth portion, when the velocity of the system is maximum of value V, is equal to

$$x. \frac{l}{n}. \frac{V}{l} = x. \frac{V}{n}$$

The kinetic energy of this portion, then, is equal to

$$\frac{1}{2}. \frac{dl}{n}\left(x \frac{V}{n}\right)^2$$

where the constant d is such that $\frac{dl}{n}$ represents the mass of the portion. Therefore, dl is the mass of the whole spring, and suppose it is equal to m. The kinetic energy of the whole spring is equal to the sum of all the energies of separate small equal parts.

Thus,

$$K.E._{total} = \tfrac{1}{2}.\ \frac{dl}{n}\left(1.\ \frac{V}{n}\right)^2 + \tfrac{1}{2}.\ \frac{dl}{n}\left(2.\ \frac{V}{n}\right)^2 +$$

$$\tfrac{1}{2}.\ \frac{dl}{n}\left(3.\ \frac{V}{n}\right)^2 + \ ..\ \ + \tfrac{1}{2}.\ \frac{dl}{n}\left(x.\ \frac{V}{n}\right)^2 +$$

$$\ldots\ldots\ldots\ldots\ldots\ldots + \tfrac{1}{2}.\ \frac{dl}{n}\left(n.\ \frac{V}{n}\right)^2$$

$$= \tfrac{1}{2}.\ \frac{dl}{n} \cdot \frac{V^2}{n^2} \cdot (1^2 + 2^2 + 3^2 + .. + x^2 + \ .. + n^2)$$

$$= \tfrac{1}{2}.\ \frac{dl}{n} \cdot \frac{V^2}{n^2} \cdot \frac{n(n-1)(2n+1)}{6}$$

When n is very large, $(n - 1)$ and $(2n + 1)$ may be approximately taken as n and $2n$ respectively.

$\therefore$ *K. E.* of the mass of the spring

$$= \tfrac{1}{2}.\ \frac{dl}{n} \cdot \frac{V^2}{n^2} \cdot \frac{2n^3}{6} = \tfrac{1}{2} \cdot \tfrac{1}{3}\, mV^2,$$

where m is the mass of the spring.

Therefore, when a vibration is set in a helical spring of mass m, due to the application of a load whose mass is M, the total kinetic energy of the system is $\frac{1}{2}\left(M + \frac{m}{3}\right)V^2$, as if the mass in motion is equal to the mass of the load plus one-third of the mass of the spring.

Illus. Ex. 82. *The amplitude of the spring vibration of a drop-valve in steam engine is 3 in. The weights of the valve-piece and the spring are 1.25 lbs. and .75 lb. respectively. The stiffness of the spring is 250 lbs. The valve travel is 1 in. Find the velocity of the valve when it strikes the seat.*

The valve travel is 1 in. and the amplitude is 3 in. Therefore, the spring is adjusted with initial 2 in. compression. e being 250 lbs., the *K. E.* when the valve strikes the seat, according to

$$K.E. = \tfrac{1}{2}\, e\,(a^2 - x^2), \text{ is } \tfrac{1}{2} \times 250\,(3^2 - 2^2) = 625 \text{ in. pounds.}$$

$$\text{Again, } \tfrac{1}{2}\, e\,(a^2 - x^2) = \tfrac{1}{2}\left(M + \frac{m}{3}\right) V^2$$

$$\therefore\ \frac{625}{12} = \tfrac{1}{2}\left(\frac{1.25}{32.2} + \tfrac{1}{3} \times \frac{3}{4 \times 32.2}\right)V^2$$

From which, $V^2 = 2240$

$\therefore \quad V = 47.2$ feet per second.

147. Change in Kinetic energy due to Impact.

Case I. Two inelastic bodies of masses m and m' having initial velocities u and u' respectively collide directly and move together with a velocity v.

Momentum before the impact = Momentum after the impact.

$$\therefore \quad mu + m'u' = (m + m')\, v$$

$$\text{or,} \quad v = \frac{mu + m'u'}{m + m'}$$

The kinetic energy before the impact $= \frac{1}{2} m u^2 + \frac{1}{2} m'u'^2$ and after the impact

$$= \tfrac{1}{2}(m + m')\, v^2 = \tfrac{1}{2}(m + m') \left(\frac{mu + m'u'}{m + m'}\right)^2$$

$\therefore$ the change

$$= \tfrac{1}{2}(m + m') \left(\frac{mu + m'u'}{m + m'}\right)^2 - \tfrac{1}{2} mu^2 - \tfrac{1}{2} m'u'^2$$

$$= -\tfrac{1}{2} \cdot \frac{mm'}{m + m'} (u - u')^2$$

That is to say, there is a loss in the system due to the impact by an amount equal to

$\frac{1}{2} \frac{m m'}{m + m'} (u - u')^2$, *i.e.*, if Q_L represents the loss of energy,

$$Q_L = \tfrac{1}{2} \cdot \frac{m\, m'}{m + m'} (u - u')^2 \qquad \text{............}\textit{Eq. } 81$$

Illus. Ex. 83. *A bullet, weighing 1 oz., is projected with a velocity of 1500 feet per second. It overtakes a block of wood, weighing 4 lbs., which moves with a velocity of 50 feet per second. If the collision be a direct one, find the loss of kinetic energy due to the impact.*

$$\text{Loss} = \tfrac{1}{2}\cdot \frac{\dfrac{1}{16 \times 32.2} \times \dfrac{4}{32.2}}{4\frac{1}{16} \div 32.2}\,(1500 - 50)^2$$

$$= 2009 \text{ ft. lbs.}$$

Case II. Two smooth elastic spherical bodies of masses m and m' having initial velocities u and u' respectively collide directly.

Let e be the coefficient of restitution and v and v' be the velocities of the bodies respectively after the impact.

Then, as before, $m\,u + m'\,u' = m\,v + m'\,v'$ $\quad$ (*i*)

Also we know that, $v - v' = -\,e\,(u - u')$ $\quad$ (*ii*)

The equation (*i*) may be put as

$$m\,u + m'u' - m\,u' + m\,u' = m\,v + m'v' - m\,v' + m\,v'$$

or, $$m\,(u - u') + u'\,(m + m') = m\,(v - v') + v'\,(m + m')$$

$$= -\,e\,m(u - u') + v'(m + m')$$

$$\therefore \quad v' = u' + \frac{m\,(1 + e)\,(u - u')}{m + m'}$$

Similarly, it can be found out that

$$v = u - \frac{m'\,(1 + e)\,(u - u')}{m + m'}$$

Therefore, the change in $K.E.$

$$= \tfrac{1}{2}\,m\left\{u - \frac{m'\,(1 + e)\,(u - u')}{m + m'}\right\}^2 +$$

$$\tfrac{1}{2}\,m'\left\{u' + \frac{m\,(1 + e)\,(u - u')}{m + m'}\right\}^2 - \tfrac{1}{2}\,mu^2 - \tfrac{1}{2}m'\,u'^2$$

$$= -\,\frac{1 - e^2}{2}\cdot\frac{mm'}{m + m'}\,(u - u')^2$$

That is to say, there is a loss due to impact by an amount

$$Q_L = \frac{1 - e^2}{2} \times \frac{m\,m'}{m + m'}\,(u - u')^2 \qquad \cdots\textit{Eq. } 82$$

It is to be marked here that while the masses are inelastic, e becomes zero, and the loss becomes equal to $\frac{1}{2}.\frac{mm'}{m+m'}(u-u')^2$ and this was proved in *Case 1*.

Illus. Ex. 84. *A spherical mass weighing 2 lbs collides directly with another spherical mass of the same material at rest. After the impact the first one stops moving and the second one moves with half the velocity of the first one. If the coefficient of restitution be ⅔, find the weight of the second one.*

Let u be the velocity with which the first ball strikes the second one, and let w be the weight of the second ball.

Now, K.E. of the *1st* − Loss due to impact = K.E. produced in the second.

$$\therefore \frac{1}{2}\cdot\frac{2}{g}u^2 - \frac{1-\left(\frac{2}{3}\right)^2}{2}\times\frac{\frac{2}{g}\cdot\frac{w}{g}}{\frac{2}{g}+\frac{w}{g}}u^2 = \frac{1}{2}\cdot\frac{w}{g}\left(\frac{u}{2}\right)^2$$

$$\text{or}\quad \frac{1}{2}u^2\left(\frac{2}{g}-\frac{5}{9}\cdot\frac{2w}{g^2}\cdot\frac{g}{2+w}\right)=\frac{1}{2}u^2\frac{w}{4g}$$

$$\text{or,}\quad \frac{36g+8gw}{9g^2(2+w)}=\frac{w}{4g}$$

or, $9w^2 - 14w - 144 = 0$ from which $w = 4.85$ lbs.

Case III. If the collision of the two spheres be an oblique one, then taking the notations as in article 119, prob. 2(b), and following the solution of the previous problem,

$$\tfrac{1}{2}mv^2\cos^2\beta + \tfrac{1}{2}m'v'^2\cos^2\beta_1 - \tfrac{1}{2}mu^2\cos^2\alpha - \tfrac{1}{2}m'u'^2\cos^2\alpha_1$$

$$= -\frac{1-e^2}{2}\cdot\frac{mm'}{m+m'}(u\cos\alpha - u'\cos\alpha_1) \qquad (i)$$

Again, $v\sin\beta = u\sin\alpha$, and $v'\sin\beta_1 = u'\sin\alpha_1$

$$\therefore\ \tfrac{1}{2}m\,v^2\sin^2\beta + \tfrac{1}{2}m'\,v'^2\sin^2\beta_1$$

$$= \tfrac{1}{2}m\,u^2\sin^2\alpha + \tfrac{1}{2}m'\,u'^2\sin^2\alpha_1 \qquad (ii)$$

Now, adding (*i*) and (*ii*),

$$\tfrac{1}{2}m\, v^2 + \tfrac{1}{2}m'\, v'^2 - \tfrac{1}{2}m\, u^2 - \tfrac{1}{2}m'\, u'^2$$

$$= -\frac{1 - e^2}{2}\,\frac{m\,m'}{m + m'}\,(u \cos\alpha - u' \cos\alpha_1)^2$$

In this case too, there is a loss due to the impact and the amount is equal to

$$Q_L = \frac{1 - e^2}{2} \cdot \frac{m\,m'}{m + m}\,(u \cos\alpha - u' \cos\alpha\ ) \qquad \textit{Eq. } 83$$

Illus. Ex. 85. *If in the example of the previous case the balls are allowed to move on a smooth horizontal plane, find the loss in K L due to impact when the velocity of the first one is 15 feet per second and the common normal makes an angle of 30° with the plane*

$$\text{Loss} = \frac{1 - (\frac{2}{3})^2}{2} \times \frac{\frac{2}{g}\ \frac{4\ 85}{g}}{\frac{2 + 4\ 85}{g}} \times (15)^2 \cos^2 30$$

$$= \frac{5}{18}\ \frac{9\ 7}{g^2}\ \frac{g}{6\ 85} \times 225 \times (\ 866)$$

$$= 2\ 06 \text{ foot pounds}$$

It is to be marked that in the Cases (*II*) and (*III*) if the bodies are perfectly elastic, *i.e.*, when $e = 1$, there is no loss due to the impact

Generally the loss in kinetic energy due to an impact appears in the shape of heat energy

PROBLEMS

142. If the working strength of a chain in a crane be 1 ton, how much work can be done at a time in raising loads on the first floor of a workshop building 20 feet above the ground level? *Ans* 44800 ft lbs

143. How many ergs are done when 100 metres cube of water is raised through 20 metres? *Ans* $1\ 96 \times 10^{18}$ ergs.

144. What fraction of the weight of a 2-ton roller should be applied by each of the labourers of the gang consisting of 5, in order to drag it 100 feet for 30000 foot-pounds of work? *Ans* 013 of the weight

145. The helmsman keeping the direction right, two men draw a boat parallel to the straight coast of a river with the help of a rope tied to the mast. If the rope makes an angle of 15° with the course and if each of the men exerts a force of 30 lbs., how much work is done when they draw the boat one mile? *Ans.* 306000 ft. lbs.

146. Find the work done in punching a ¾ inch hole in a ¾ inch steel plate. The ultimate strength of the plate against shearing is 40000 lbs. per sq. in. Assume that the force required to punch varies directly as the thickness of the plate. *Ans.* 2208 ft. lbs.

147. A man rotates a handle at the circumference of a wheel 20 in. in diameter which is rigidly fixed with a drum, with a force of 20 lbs. The system is used as a lifting machine What is the work done to lift a load if 50 turns of the handle is required? *Ans.* 5236 ft. lbs.

148. A load of 1000 kg. is lifted through 5 metres with the help of a hand-driven lifting machine. If 4 turns of the driving wheel can do the job, find the average torque. *Ans.* 1.953×10^{10} dyne-centimetres.

149. Solve the problem No. 135 with the help of the energy equation.

150. The steering wheel of a motor car is 18 inches in diameter. The driver of the car turns the wheel through 90° exerting a tangential force of 11 lbs. by each hand at the circumference of the wheel Find the work done.
Ans. 310.86 in. lbs.

151. A line-shaft in a workshop rotates at a speed of 120 r. p. m. If the difference in tensions of the tight and loose sides of the belt about the driven pulley, 60 in. diameter, be 225 lbs., find how much work is being done by the line-shaft per minute. *Ans.* 423900 ft. lbs.

152. Find the horse-power of an automobile engine when it runs at a constant speed of 30 miles per hour against a road resistance of 100 lbs.
Ans. 8 H.P.

153. A motor car weighing 10 cwt. attains a speed of 30 miles per hour in 10 seconds. Find the H.P. of the engine at that speed if the road and other resistances amount to 20 lbs. per ton weight of the car.
Ans. 13.048 H.P.

154. A lorry weighing 2 tons runs up an inclined plane 1 in 200 with a speed of 20 m.p.h. Find with what H.P. the engine is working if the friction and air resistances against the motion be 20 lbs. per ton.
Ans. 3.328 H.P.

155. The engine of a steamer develops 150 H.P. and runs at a constant speed of 12 miles per hour. Find the resistance offered by air and water.
Ans. 4687.5 lbs.

156. When the speed of a ship is constant at v miles per hour, the horse-power maintained by the engine is 5000. Find v if the resistance offered by air and water is 62500 lbs. *Ans.* 30 miles per hour.

157. Find the work done in filling an overhead tank $20' \times 10' \times 10'$, placed at a height of 100 feet, from a well where the surface of water remains constant at 20 feet below the ground level. The 100 feet height means the rise of water from the ground level. *Ans.* 14952000 ft. lbs.

158. If the pumping set in the previous problem fills the tank in 2 hours, and if the machine is 70% efficient, find the horse-power of the machine. *Ans.* 5.4 H.P.

159. In a generating plant the motive power is water at a high pressure of 3000 lbs. per square inch. Determine the feed of water in cu. ft. per hour for a supply of 576 horse-power. If there is a loss of power by 25% due to various causes, find the consumption for the same supply. *Ans.* 3520 cu. ft.

160. A portion of a river current, 1000 sq. ft. in cross-section with a flow speed of 3 miles per hour, is arranged to be utilised for a turbine. If the turbine is 75% efficient and if the fall available at its vicinity is 20 feet, find the horse-power that can be developed by the machine. *Ans.* 7500 H.P.

161. A Belis-Morcom vertical type steam engine can develop 75 H.P. with 500 r.p.m. What is the torque produced in the fly-wheel shaft? *Ans.* 788.3 lb. ft.

162. Net force to rotate a pulley 15 in. in diameter on a line-shaft of a workshop increases from 100 to 150 lbs., while the shaft rotates uniformly at 100 r.p.m. Find the increased H.P. supplied. *Ans.* .5947 H.P.

163. In a tube-well water always remains at 15 feet below the earth surface. An electrical motor-driven pumping set is fitted to fill an overhead reservoir of 20000 gallons capacity at a height of 75 feet. If the pump can fill it in one hour, select the motor for the purpose. One gallon of water weighs 10 lbs. *Ans.* 9.09 H.P. motor.

164. A train weighing 200 tons starts from rest. The draw-bar pull is 4 tons and the resistances amount to 12 lbs. per ton. Find the K.E. of the engine after 1 min. 35 secs. *Ans.* 13960000 ft. lbs.

165. A train weighing 150 tons and running at a speed of 30 m.p.h. comes at the foot of an inclined plane and the steam is shut off. If the inclination of the plane be 5°, calculate the drop in kinetic energy after it has run through 100 yards. *Ans.* 878400 ft. lbs.

166. The velocity of a train weighing 200 tons is developed from rest to 45 m.p.h. in 2 minutes, during which it travels 1000 yards. If the resistances amount to 10 lbs. per ton, find the average H.P. *Ans.* 549.5 H.P.

167. After 15 seconds from starting the H.P. of the engine of a motor car weighing 11 cwt. is found to be 10. The road friction offers a resistance of 20 lbs. per ton weight. If the car travels 100 yds. in that time, find the velocity developed by the car. *Ans.* 44 miles per hour.

168. Steam is issuing in parallel streams from a jet of .25 in. diameter with a velocity of 3000 feet per second for driving a steam turbine. If the density of steam be such that one pound of steam occupies 25 cu. ft., find the H.P. of the jet. *Ans.* 10.39 H.P.

169. What fraction of a pound's weight acting on a mass weighing 64.4 lbs. through a distance of 10 feet in the direction of its motion, will double the K.E. of the mass, if the initial velocity before the force being applied be 180 feet per minute? *Ans.* .9.

170. What is the K.E. of a bullet weighing 1 oz. when it is fired at a velocity of 1500 feet per second? If the said bullet strikes a block of wood weighing 5 lbs. resting on a perfectly smooth plane and carries the block with it, find the loss in K. E. due to impact.

Ans. 2184 ft. lbs., 2157 ft. lbs.

171. In altering the speed of a fly-wheel from 99 to 100 r.p.m. the K. E. changes by an amount of 300000 ft lbs. What is the moment of inertia? What is the K. E. of the wheel when it makes 120 r.p.m.?

Ans. 275000 gravitational units, 21710000 ft. lbs.

172. If the mass of a fly-wheel weighing 1000 kg. can be considered to be concentrated at a distance of 1 metre from the axis, find the K. E. of the wheel when it rotates at a speed of 200 r.p.m.

Ans. 6085000 ergs.

173. Steam leaves a nozzle and enters the blades of a steam turbine at a speed of 2500 ft. per second. The rate of discharge is .75 lbs. per second. Find the torque exerted on the shaft if all the kinetic energy of the steam is converted into work. The speed of the motor carrying the blades is 2000 r.p.m. *Ans.* 347.7 lb. ft.

174. A single acting steam engine has a speed of 300 r.p.m. The piston stroke is 12 inches. In a laboratory experiment indicator diagram for the cylinder is taken. The acquired area is measured by a planimeter and found to be 2.68 sq. inches. The linear scale for the card is $1'' = 3''$, and the force scale is $1'' = 75$ lbs. Find the work done in one minute if the diameter of the piston is 6 inches. *Ans.* 426000 ft. lbs.

175. If the engine of the previous problem be double acting and if the piston area be taken the same for both the sides, find the work done per minute. *Ans.* 852000 ft. lbs.

176. If the cross-sectional area of the piston rod of the engine of the problem No. 175 be 3.3% of the piston area, find the work done per minute.

Ans. 831010 ft. lbs.

177. In the illustrated example 79, find the distance moved by the load, 1200 lbs., when it acquires a velocity of 30 ft. per sec.

Ans. 223.6 feet.

178. If in the illustrated example 79 a resistance acts against the motion of the system (which is a sure factor in each mechanism) by an amount equal to 10 lbs., find the velocity of the smaller mass.

Ans. 12 feet per second.

179. The stiffness of the spring in a spring gun is such that one pound is required for an alteration of one inch from the free length of the spring. The spring is compressed by 5 inches and a metal ball weighing one ounce is placed at the compression head of the spring and the spring is released; find with what velocity the ball will leave the gun.

Ans. 32.76 feet per second.

180. The stiffness of a helioal spring is such that a force of 1010 lbs. is required to compress it by 1 inch. The axis of the spring is kept vertical and is compressed by 4 inches. A mass of 20 lbs. is placed at the head of the compressed spring and the spring is released. Determine the kinetic energy of the mass when the spring regains its free length. Neglect the consideration of the mass of the spring and the air resistance. Also determine the height to which the mass rises from its original position of rest.

Ans. 666.66 ft. lbs., 400 inches.

181. The stiffness of a helical spring is such that 4000 lbs. can compress the spring by 1 inch. A body weighing 50 lbs. falls freely on the head of the spring kept vertical and compresses it by 6 inches. What is the maximum kinetic energy of the mass? Determine through what vertical distance the body will fall before attaining the zero velocity.

Ans. 5975 ft. lbs., 120 ft.

182. A loose wagon weighing 15 tons runs into a spring buffer at the end of a siding with a velocity of 5 feet per second. The stiffness of the spring requires 30000 lbs. to compress it by 1 inch. Compute the maximum compression of the spring due to the impact. *Ans.* 3.229 inches.

183. The table of a machine weighing 150 lbs. is arranged to move in S.H.M. horizontally. If the travel of the part be 18 inches and the mean speed is 100 feet per minute, find the force due to inertia at the end of the stroke, and also find the kinetic energy at the one-third stroke.

Ans. 68.87 lbs., 23.84 ft. lbs.

184. The maximum velocity, *i.e.*, the velocity at the lowest point of the path of motion of a simple pendulum, whose bob weighs 2 lbs., is 4 feet per second. If the pendulum makes one oscillation per second, and if the loss of energy due to air, friction and other resistances is .00005 ft. lb. per second, find after what time the pendulum will cease to oscillate.

Ans. 2 hrs. — 45 min. — 40 secs.

185. The adjusting pressure on the spring of a drop valve-set in a steam engine is 300 lbs. If the stiffness of the spring be 200 lbs., find the amplitude of the spring vibration if the valve-travel is 1 in. What is the K. E. of the valve when it strikes the seat? *Ans.* 2.5 inches, 400 in. lbs.

186. Two boys are playing with marbles weighing .25 oz. each. One of them strikes the other's ball at rest directly with his ball at a velocity of 30 ft. per sec. If the loss of K. E. due to impact be 3.22 ft. poundals, calculate the value of coefficient of restitution. *Ans.* .2899.

187. On a smooth table two balls of the same material weighing 5 oz. run from 15° west of north and 20° east of north respectively and collide. If the magnitudes of the velocities of the balls be the same as 16 ft. per sec., find the loss in K. E. due to the impact. The coefficient of restitution is .5. *Ans.* .1015 foot poundals.

PART III

CHAPTER VII

STATICS

148. Statics is the portion of Mechanics that deals with the forces in equilibrium.

The action of a force is to change the state of motion in a body. Statics deals with the forces the actions of which are neutralised by each other and no change is produced in the state of motion in a body on which they act, and this is the general condition of balanced system of forces.

149. Force—a vector quantity.

A force has a magnitude as well as direction and can, therefore, be represented by a vector drawn from the point of application of the force.

A straight line drawn through the point of application along which the direction of a force can be represented is called the *line of action* of that force.

150. The forces may have a common plane of action as well as they may act in different planes. In the former case the forces are said to be *co-planer*. Again, the forces may be *concurrent*, *i.e.*, their lines of action may meet at a point, and also they may be *non-concurrent*, *i.e.*, their lines of action have no common point of meeting.

151. In Kinematics and Kinetics the subject was treated with respect to a particle, and also it is to be noted that the motions and forces that were attributed to the particle were all co-planer. It is clear that if two of the velocities or forces be considered at a time the question of planes vanishes and the problems become purely two-dimensional. It is also evident from the definition of a rigid body that if an elementary particle in a rigid body gets a motion or displacement in a definite direction then each and every particle in it will have the same motion or displacement, *i.e.*, the body as a whole may be said to have the same motion or displacement. A body, though a three dimensional

object, is looked upon as an object of two dimensions in those cases and the problems are solved accordingly. The special treatment with rigid bodies, *i.e.*, the treatment with full consideration of three dimensions is required when the question of forces acting at points or particles in different planes arises. The problems of this nature are required to be treated with reference to three co-ordinate axes at right angles to each other.

152. The different system of forces will be discussed in the following classified order :

1. Co-planer Concurrent Forces
2. Co-planer Non-Concurrent Forces
   (*a*) Parallel Forces
   (*b*) General
3. Parallel Forces in different planes
4. Concurrent and Non-Concurrent Forces in general in different planes.

CO-PLANER CONCURRENT FORCES

153. *If the lines of action of several forces meet at a point the forces are said to be concurrent.* The point of application of a force may be taken anywhere on its line of action without hampering in any way its effect on a rigid body. Therefore, the cases of concurrent forces acting on different points of a rigid body and the forces acting on a single particle may be taken as similar in action. The forces acting on a particle is a particular case of concurrent forces where they happen to have a common point of application. Different cases of forces acting on a particle in equilibrium will be discussed here in detail. The conditions for equilibrium thus established may, therefore, be accepted as true for each and every case of concurrent forces.

154. **Convention of signs.** The forces will always be referred to the rectangular co-ordinate axes OX and OY, passing through the point representing the particle on which the forces act. The general convention of choosing the signs is that the right of Y-axis and up the X-axis are taken as positive, while the other two are taken as negative.

155. Resultant and Equilibrant. If four forces F_1, F_2, F_3, and F_4 act on a particle (Fig. 77), then, by adding the four forces vectorially the resultant *ae*, *i.e.*, R is obtained. If in addition to these four forces another force F_5 (which is equal and opposite to R) act on the particle it will be found that in drawing the vector diagram the fifth vector *ea*, representing F_5, (*i.e.* $-R$), just ends at the point *a*, leaving zero resultant.

The force represented by *ea*, which is equal in magnitude but opposite in direction to *ae* (the resultant R, of the four forces F_1, F_2, F_3 and F_4) is called the *equilibrant* of these four forces. The function of the equilibrant is to maintain equilibrium against the action of several other forces of which it is the equilibrant.

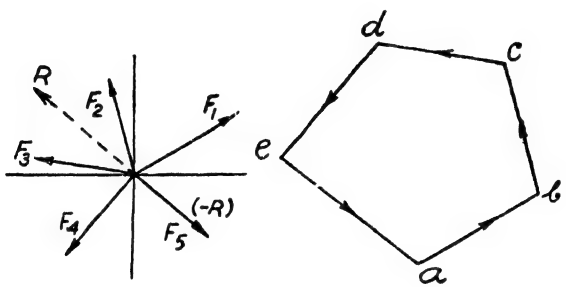


FIG. 77

156. It is evident from the above article that the vector diagram of several forces keeping a particle at rest must be a closed figure, or in other words, the geometric sum of the forces is equal to zero. The directions given by the arrow-heads are in cyclic order, and, therefore, any of the vectors may be said to represent the equilibrant of the other remaining forces.

In case, where the number of sides in the diagram is more than three, the diagram is called a *vector polygon* and where the number is three, the diagram is called a *vector triangle*.

As three forces in equilibrium form a vector triangle it can be stated as follows :

If three forces keep a particle in equilibrium they can be represented in magnitudes and directions by the sides of a triangle taken in order.

157. Methods of determining the Equilibrant. It is advisable to recall here the subject matter discussed in article 21. It is shown there that if several forces act upon a particle, the resultant force can be found out by the method of vector addition, the method of resolution and composition and also from the trigonometrical relations. Here, of course, the discussion is restricted to only those cases where the resultant becomes zero, so that there is no change in the state of motion in the body.

(*I*) *Vector diagram.* By drawing the vectors in order, one after another, and finally closing the diagram. The closing vector is the required force or the equilibrant. The directions, of course, will be in cyclic order.

Note that the vector diagram can also be drawn if any two elements of two consecutive forces are unknown. The two elements may be either two magnitudes or two directions or one magnitude and one direction. The third element, point of application, is known in this case.

(*II*) *The method of resolution and composition.* It is also similar to the case of finding out the resultant of several unbalanced forces. The only difference is that the direction will be just opposite to that of the resultant. In cases of forces in equilibrium, therefore, the sums of the vertical and horizontal components must individually be equal to zero.

(*III*) *In case of two unbalanced forces the equilibrant can be found out with the help of*

(*a*) Triangle of forces

(*b*) Parallelogram of forces

In both the cases if R be the resultant, $R = \sqrt{P^2 + Q^2 + 2PQ \cos \theta}$, where P and Q are the two component forces and θ is the angle between them. The equilibrant is just equal and opposite to the resultant.

(*c*) Lami's Theorem.

If three forces keep a particle in equilibrium each of the forces is proportional to the sine of the angle between the other two.

Let P, Q and R be the three forces in equilibrium acting at O (Fig. 78). The vector triangle abc is drawn—ab representing P, bc representing Q and ca representing R in magnitude and direction. In a triangle the sides are proportional to the sines of the opposite angles. Therefore, in the triangle abc

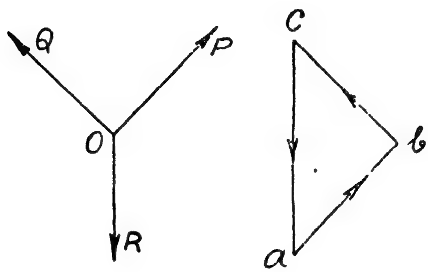


FIG. 78

$$\frac{ab}{\sin acb} = \frac{bc}{\sin cab} = \frac{ca}{\sin abc} \qquad (i)$$

But,

$$\angle acb = 180 - \angle QOR$$

$$\angle cab = 180 - \angle POR$$

$$\angle abc = 180 - \angle POQ$$

Therefore, from the relations (i),

$$\frac{ab}{\sin QOR} = \frac{bc}{\sin POR} = \frac{ca}{\sin POQ} \qquad (ii)$$

As the sides of the triangle are drawn proportional to the forces,

$$\frac{P}{ab} = \frac{Q}{bc} = \frac{R}{ca} \qquad (iii)$$

Now, multiplying (ii) by (iii),

$$\frac{P}{\sin QOR} = \frac{Q}{\sin POR} = \frac{R}{\sin POQ} \qquad \ldots\ldots\ldots\ldots Eq.\ 84$$

(*IV*) *The principle of moments* which is explained in the following articles.

158. Moment of a force. If a force P, acting on a particle, tends to rotate it about O (Fig. 79), the moment is equal to $P.r$, where r is the perpendicular distance of the line of action of the force from O. Let P be represented by the vector ab, then, $P.r = ab.\ r =$ twice the area of the triangle abO.

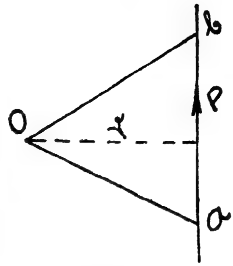


FIG. 79

159. *Moment of the resultant force is equal to the algebraic sum of the moments of the component forces about any point in their plane of action.*

Let two forces P and Q act on a particle at a (Fig. 80). Take any point O in the plane of the two forces. From O draw a straight line parallel to the line of action of the force P cutting the line of action of the force Q at d. Choose a scale so that ad represents the force Q. In the same scale measure ab to represent the force P. Then, $ab = ad \times \frac{P}{Q}$. Complete the parallelogram $abcd$. The diagonal ac must represent the resultant of P and Q. Join Oa and Ob.

The moments of P and Q about O are represented by twice the areas Oab and Oad respectively. But $\triangle\, Oab = \triangle\, abc = \triangle\, acd$. Therefore, $\triangle\, Oab + \triangle\, Oad = \triangle\, acd + \triangle\, Oad = \triangle\, Oac$, which represents half the moment of the resultant of P and Q about O.

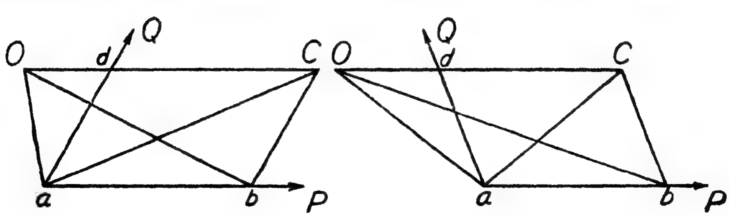


FIG. 80

Hence, the sum of the moments of P and Q about any point O in the plane of action of the forces is equal to the moment of their

resultant about the same point. *That is, the sum of the moments of P and Q and their equilibrant (which is equal in magnitude but opposite in direction to their resultant) about any point in their plane of action must be zero.* The idea can be extended to any number of forces taking two at the first instance, then with their resultant a third force should be considered, and so on. Thus, the moment of the resultant force about any point in the plane of the forces is equal to the algebraic sum of the moments of the component forces about the same point. If an axis be conceived through the point O which is at right angles to the plane of action of the forces, then also, it is evident that the moment of the resultant about the axis, *i.e.*, about O is equal to the algebraic sum of the moments of the component forces about the same axis.

160. Conditions of Equilibrium of Forces acting on a Particle.

1. The vector diagram will be a closed figure. That is, the Geometric sum of the forces is zero. This is called the *Graphical condition* of equilibrium.

2. If the forces be resolved along two rectangular co-ordinate axes, OX and OY, one being horizontal and the other vertical, each of the sums of the vertical and horizontal components must individually be equal to zero, *i.e.*,

$$\Sigma H = 0 \qquad \Sigma X = 0$$

i.e.,

$$\Sigma V = 0 \qquad \Sigma Y = 0$$

This is called the *Algebraic condition* of equilibrium.

3. The algebraic sum of the moments of the forces about any point in their plane of action or about any axis perpendicular to the plane must be zero. In other words, $\Sigma M = 0$, where ΣM represents the sum of the moments.

It is to be noted that the point about which the moments are to be found out must not be on the line of action of the resultant. Because, though a number of forces may not be in equilibrium, the sum of the moments about a point in the line of action of the resultant must be zero as the moment of the resultant about that point is zero. Therefore, to become definite, moments about any three points not in a straight line are to be found out and in each case the sum to be equated to

zero. Any two of the points may be on the line of action of the resultant but the third point, then, must be outside that line. If the sum of the moments about the third point be zero, it assures that there is no resultant of the forces, which indicates that there is no motion of translation in the body. The sum of the moments being zero, it also assures that there is no turning tendency, *i.e.*, motion of rotation in the body.

The first two of the conditions only state about the motion of translation of the body.

It is to be marked here that in case of concurrent forces in equilibrium, if any of the three conditions is satisfied, the other two will automatically be fulfilled.

It is evident that more than *two* unknown elements in such a force system cannot be determined from the conditions stated above.

Conclusion. If several forces act on a particle, either they will be in equilibrium, or they shall have a resultant, *i.e.*, they can be replaced by a single force.

161. *If three forces are in equilibrium and if any two of them meet at a point the third one must pass through their point of intersection.*

The two intersecting forces have a resultant, the line of action of which must contain the point of intersection and only an equal and opposite force can neutralise its action. Therefore, the line of action of the third force which neutralises the effect of the resultant to keep the body at rest, must be along the same line with the resultant, *i.e.*, must pass through the point of intersection.

Illus. Ex. 86. *A string 14 feet long is fixed at its ends with a horizontal bar at a distance of 10 feet. A weight of 50 lbs. is suspended from a point in the string at a distance of $\frac{3}{7}$ of the length from one end. Find the tensions in the two portions of the string.*

Let A and B be two points in the bar at a distance of 10 ft. (Fig. 81). The ends of the string ACB are fixed at A and B. The point C is such that $AC = \frac{3}{7}.14 = 6$ ft. and therefore, $CB = 8$ ft. Again, because $6^2 + 8^2 = 10^2$, the angle ACB is a right angle.

Construction. Take any point a. From a draw a vertical line ab in the same direction with the suspended weight and measure ab equal to 2.5 in. according to the scale chosen to represent 50 lbs. Through b and a draw bc

and *ac* parallel respectively to *CB* and *CA* cutting at *c*. Then *bc* and *ca* represent the tensions in *CB* and *CA*. Measure *bc* and *ca*, which are equal to 1.5 ins and 2 ins respectively. Therefore, the tensions are 1.5×20 and 2×20 lbs, *i e*, 30 lbs and 40 lbs respectively.

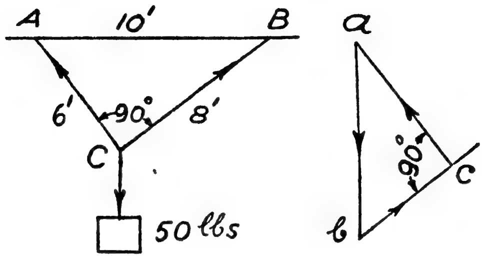


FIG. 81
Scale 1" = 20 lbs

Illus. Ex. 87. *The resultant of two equal forces is a third equal force. Find the angle between the first two forces.*

(*I*) *From the idea of the vector diagram.* From the question it is evident that the vector diagram formed by the two forces and their resultant will be an equilateral triangle. Each of the angle of this triangle is 60°. Therefore, the angle between the two forces in question must be $(180° - 60°) = 120°$

or

(*II*) *From trigonometrical relation.* $R^2 = P^2 + Q^2 + 2\,PQ \cos\theta$. Here, P, Q and R are equal,

$$\therefore\ P^2 = P^2 + P^2 + 2\,P\,P \cos\theta$$

$$= 2\,P^2 + 2\,P^2 \cos\theta$$

i e, $\cos\theta = -\frac{1}{2}$ $\quad \theta = 60°$, when $\cos\theta = \frac{1}{2}$

The minus ($-$) sign indicates that it must either be in the second or the third quadrant. Therefore, the angle between the two forces is $(180° - 60°) = 120°$

or

(*III*) *From Lami's theorem.* Because the resultant and the equilibrant are equal and opposite, if R be the resultant of the two forces P and Q then,

$$\frac{P}{\sin QOR} = \frac{Q}{\sin POR} = \frac{R}{\sin POQ}$$

But, $P = Q = R$ (in magnitude). Therefore, $QOR = POR = POQ$.

The sum of these three angles must be 360° and hence each of the angles, is equal to 120°.

Illus. Ex. 88. *Forces acting at a point are represented in magnitude and direction by 2 AB, 3 BC, 2 CD, DA, CA and DB, where ABCD is a quadrilateral (Fig 82) Show that the forces are in equilibrium*

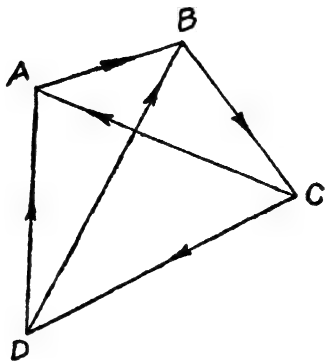


FIG 82

1 $\quad AB + BC = AC, \quad \therefore \quad 2AB + 2BC = 2AC$

2 $\quad BC + CD = BD$

3 $\quad CD + DA = CA$

By addition, $2AB + 3BC + 2CD + DA = 2AC + BD + CA$

Add DB and CA with the quantities on both the sides of the equal sign, then,

$$2AB + 3BC + 2CD + DA + CA + DB = 2AC + BD + CA + DB + CA$$

or

The resultant of AB and BC is AC

$\therefore$,, ,, $2AB$ and $2BC$ is $2AC$

Again ,, ,, BC and CD is BD

Also ,, ,, CD and DA is CA

$\therefore$,, ,, $2AB$, $2BC$, BC, CD, CD and DA is $2AC$, BD and CA

$\therefore$,, ,, $2AB$, $3BC$, $2CD$, DA, DB and CA is $2AC$, BD, CA, CA and DB

i.e., $2AC$, $2CA$, BD and DB

i.e., O.

Illus. Ex. 89. *A small ring is placed at the centre of a regular hexagon and kept in position by six strings drawn tight, all in the same plane of the figure, and each fastened at the other end to an angular point of the hexagon. The tensions in the four consecutive strings are 2, 7, 9* and *6 lb. weight respectively. Find the tensions in the two remaining strings.*

Method. Vector diagram. As the forces are in equilibrium they will form a closed figure. The strings are 60° apart.

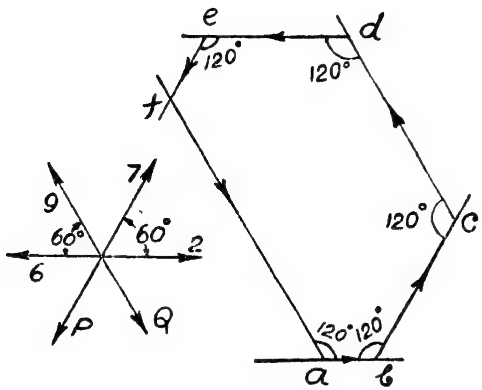


FIG. 83

Scale chosen 1′ = 2 lbs.

Construction. Take any point *a* (Fig. 83). From *a* draw any straight line *ab*, 1 in. long to represent the first force 2 lbs. in the scale chosen. From *b* draw *bc* making an angle of 120° with *ab*. Measure *bc* = 3.5 ins. to represent the force in the second string, 7 lbs. From *c* again draw *cd*, 4.5 ins. long, making an angle of 120° with *bc* to represent the third force, 9 lbs. In the same way draw *de*, 3 ins. long making an angle of 120° with *cd* to represent the fourth force, 6 lbs. Through *e* and *a* draw two straight lines making an angle of 120° with *de* and *ba* respectively to cut at *f*. Measure *ef* and *fa*. They are equal to 1.5 and 6.5 ins. respectively. Therefore, the tensions are 3 lbs. and 13 lbs. respectively.

or

Method. Resolution and Composition.

If the forces be resolved along a direction parallel to that of the force, 2 lbs. or 6 lbs., and in the direction at right angles to it, because the forces are in equilibrium,

$$\Sigma H = 2 + 7 \cos 60 - 9 \cos 60 - 6 - P \cos 60 + Q \cos 60 = 0$$

$$\text{or,} \quad 2 + 3.5 - 4.5 - 6 - .5P + .5Q = 0$$

$$\text{or,} \quad Q - P = 10 \qquad \ldots\ldots \quad (i)$$

and $\Sigma V = 7 \sin 60 + 9 \sin 60 - P \sin 60 - Q \sin 60 = 0$

or $6.062 + 7.794 - .866P - .866Q = 0$

$$\text{or, } Q + P = 16 \qquad . \quad (ii)$$

Adding (i) and (ii),

$$2Q = 26, \quad \therefore \quad Q = 13 \text{ lbs,}$$

$$\text{and} \quad \therefore \quad P = 3 \text{ lbs.}$$

Illus. Ex. 90. *A steel roof-frame consists of 5 members all in one plane and meeting at a point. One is a tie-bar 5° down the horizontal direction and carries a tension of 40 tons; the next also is a tie-bar inclined 30° up the horizontal and sustains a pull of 30 tons; the next in continuous order, is vertical and runs upward from the joint and carries a thrust of 15 tons; and the remaining two in the same order radiate at an angle of 155°* and *190° to the first bar. Find the stresses in the last two bars and state whether they pull or push at the common joint.*

Method. Vector diagram.

First draw the positions of the members of the frame. Let OA, OB, OC, OD and OE (Fig. 84) be the five members. Adjust the arrow-heads to show the directions of the forces acting in them.

Construction. Take any point a. From a draw ab in the scale chosen to represent the force, 40 tons, along OA, and from b draw bc to represent the force, 30 tons, along OB, in magnitude and direction. In the same way draw cd to represent the force, 15 tons, along CO. Through d and a draw two straight lines parallel to OD and OE respectively cutting each other at e.

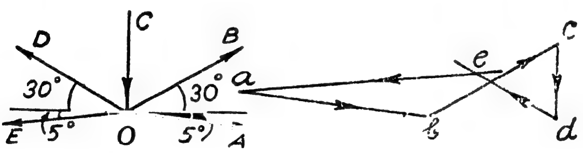


FIG. 84

Put on the arrow-heads on the sides of the figure, which of course will be in cyclic order, because the system is in equilibrium. From the directions in the diagram it is found that the last two members are in tension. Measure de and ea and multiply them by the scale. If the scale chosen be 1 in. = 10 tons,

$de = 1.6''$ and therefore, represents 16 tons.

$ea = 2.99''$ „ „ „ 29.9 tons.

Illus. Ex. 91. *A roof-truss ABC of 30 ft. span (Fig. 85) rests on two vertical supports. Considering the consequence of the variation of atmospheric*

temperature, the end A is fitted free and the end B is made fixed. If the wind pressure results a force, P = 4000 lbs., which acts at right angles to the frame member AC at its middle point and if AC = BC = 16 ft., find the reactions R_A and R_B due to this wind pressure.

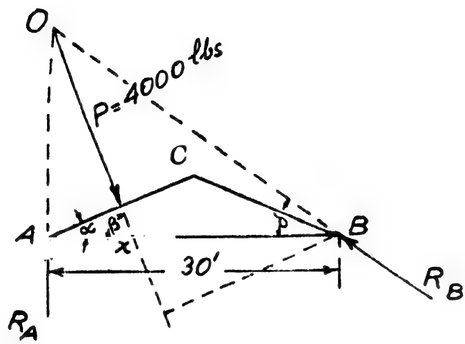


FIG. 85

Method. Principle of Moments.

$$\cos \alpha = \frac{15}{16} = .9376$$

$\therefore \quad \alpha = 20.35°$ and $\beta = (90° - 20.35°) = 69.65°$

$$Ax = 8 \div \cos 20.35 = 8.533 \text{ ft.}$$

$\therefore \quad Bx = 30 \text{ ft} - 8.533 \text{ ft} = 21.467$ feet.

At the free end A the reaction R_A must act normally and cut the line of action of P at the point O. Therefore, the reaction R_B will meet at the same point because the three forces are in equilibrium.

The moment of R_B about B is *zero*. Taking the moments of the forces about B,

$$R_A \times 30 = 4000 \times Bx \sin \beta = 4000 \times 21.467 \times .9376$$

$$R_A = \frac{4000 \times 21.467 \times .9376}{30} = 2683 \text{ lbs.}$$

Now, the magnitude of R_B may be determined in the following three different methods:—

1. $\Sigma V = O \quad \& \quad \Sigma H = O$

$$\Sigma V = R_A - 4000 \sin \beta + V_B = O$$

$$V_B = -2683 + 4000 \times .9376 = 1067.4 \text{ lbs.}$$

$$\Sigma H = 4000 \cos \beta + H_B = O$$

$$H_B = -4000 \times .348 = 1392 \text{ lbs.}$$

$\therefore \quad R_B = \sqrt{(1067.4)^2 + (1392)^2} = 1756$ lbs.

2. $\quad \tan \delta = \dfrac{AO}{AB} = \dfrac{AO}{30}$

But, $AO = Ax \tan \delta = 8.533 \times 2.689 = 22.95$ feet

$\therefore \quad \tan \delta = 22.95 \div 30 = .765$

$\therefore \quad \delta = 37° - 24' \quad \& \quad \sin \delta = .6074$

Now, taking moments about A,

$4000 \times 8 = R_B \times 30 \sin \delta = R_B \times 30 \times .6074$

Whence, $R_B = 1756$ lbs.

3. Vector addition (by drawing the actual vector diagram or with the help of trigonometrical ratio).

Choose a scale, $1'' = 1000$ lbs. and draw the vector diagram as shown in Fig. 86. The vector $ab = 2683$ lbs., $bc = 4000$ lbs. The vector ca will represent the reaction R_B. ca measures 1.75 inches approximately, and, therefore, represents 1750 lbs. approximately.

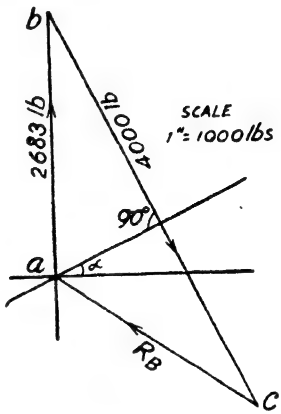


FIG. 86

CO-PLANER PARALLEL FORCES

162. When the lines of action of the forces are parallel the forces are said to be parallel forces. From the definition it is clear that the forces do not meet at a point. The case of parallel forces is a particular instance of non-concurrent forces. Parallel forces may be classified into two different groups :

(I) Like Parallel Forces, *i.e.*, when the directions of the forces are the same, and

(II) Unlike Parallel Forces, *i.e.*, when the directions are opposite.

The discussions will be proceeded with two forces only. If the cases of two forces are understood clearly, the cases with a number of forces can be easily dealt with.

163. In case of unlike parallel forces, the forces in two definite directions are counted in two opposite senses—positive and negative.

164. Resultant and Equilibrant of two like parallel forces.

Let two like parallel forces, P_1 and P_2 (Fig. 87), act on two points A and B of a rigid body respectively. Join AB and introduce two equal but opposite forces Aa and Bb at A and B respectively as shown in the diagram. As Aa and Bb are acting along the same straight line and are equal and opposite in direction, they cannot disturb the condition of equilibrium of the system. Aa and P_1 give the resultant Ac and Bb and P_2 give the resultant Bd. Produce cA and dB to meet at O. As the point of application can be taken anywhere on the line of action of a force, let O be the point of application of the two

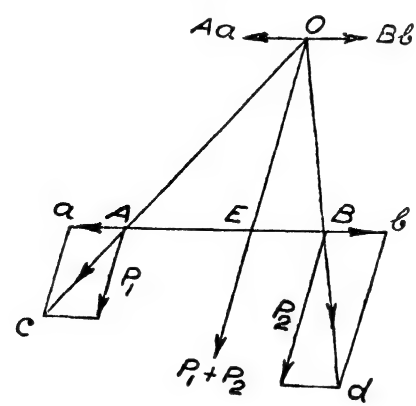


FIG. 87

resultant forces. Resolve the two resultant forces at O into their previous components. The resultant of Aa and Bb at O being zero a force of the amount of $(P_1 + P_2)$ acting in the same direction with P_1 and P_2 along OE is left, which is nothing but the resultant of P_1 and P_2. If a force equal to $P_1 + P_2$ acts along OE just in the opposite direction to P_1 and P_2, the force will balance the action of P_1 and P_2. That force is the *equilibrant* of the forces P_1 and P_2.

165. Two unlike parallel forces.

In this case let P_1 be greater than P_2. Then the two resultants, obtained in the same way as in the previous case, will meet at O as

shown in the diagram (Fig. 88). The resultant in this case is $(P_1 - P_2)$ along EO, which is parallel and in the same direction with

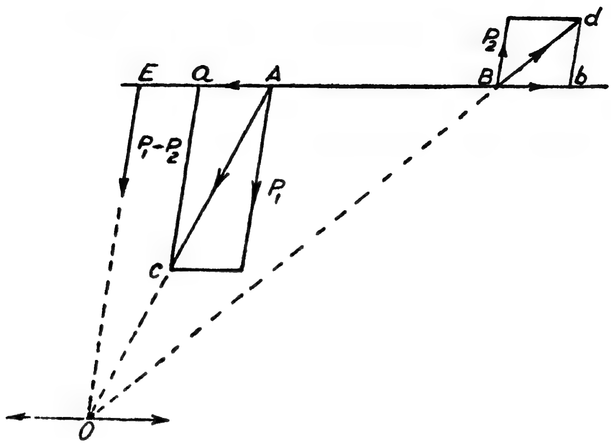


FIG. 88

P_1, the greater force. Hence, the equilibrant in this case is of the same amount with the resultant and will act just in the opposite direction to the resultant at E.

Thus, the magnitude of the resultant or equilibrant, in both the cases, is equal to the algebraic sum of the magnitudes of the component forces. The direction in the former case is the same with that of the forces; whereas, in the latter case it is the same with that of the bigger force.

166. In both the cases of articles 164 and 165 from the similarity of triangles between Aac and AEO and Bbd and BEO (Figs. 87 & 88),

$$\frac{Aa}{ac} = \frac{AE}{EO} \quad \ldots\ldots(i) \quad \text{and,} \quad \frac{Bb}{bd} = \frac{BE}{EO} \quad \ldots\ldots(ii)$$

Aa being equal to Bb, dividing (i) by (ii),

$$\frac{bd}{ac}, \text{ i.e., } \frac{P_2}{P_1} = \frac{AE}{BE} \quad \ldots\ldots\ldots Eq.\ 85$$

Again,

$$\frac{P_2}{P_1 + P_2} = \frac{AE}{AE + BE} = \frac{AE}{AB} \text{ (fig. 87)} \quad \ldots\ldots\ Eq.\ 86$$

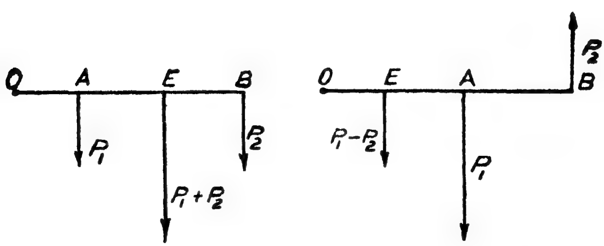
And,

$$\frac{P_2}{P_1 - P_2} = \frac{AE}{BE - AE} = \frac{AE}{AB} \text{ (fig. 88)} \qquad \cdot \qquad Eq.\ 87$$

It is obvious from the *Equation* 77 that the line of action of the resultant or the equilibrant divides the line AB in Fig. 87 internally and in Fig. 88 externally in the inverse ratio of the forces. It is to be marked that if the points of application remain the same but the directions of the forces are turned by the same amount in the same sense, the point E remains unchanged in position. It is also clear that if the magnitudes of the forces be proportionally changed the position of the point E will not be changed if the points of application remain unchanged. Again, any straight line joining any two points on the lines of action of the two forces P_1 and P_2 representing the two points of application respectively is divided in the same ratio by the point E. The point E is called the *centre of parallel forces* of that system. If the two points of application are definite, the point E is also definite.

167. *Algebraic sum of the moments of two parallel forces about any point in the plane through the lines of action of the forces or any axis perpendicular to the plane is equal to the moment of their resultant about the same point or the axis.*

Take a point O on the plane as mentioned (Fig 89), which may also be considered as the trace of an axis perpendicular to the plane of action of the forces on the plane.



A (like) FIG 89 B (unlike)

From O draw a straight line cutting the lines of action of the forces P_1 and P_2 and their resultant at right angles. It has been proved

that $\dfrac{P_1}{P_2} = \dfrac{AE}{BE}$

i.e., $P_1 \times AE = P_2 \times BE$

In Fig. 89-*A*, the sum of the moments of the component forces about

$$O = (P_1 \times OA) + (P_2 \times OB) = P_1(OE - EA) + P_2(OE + EB)$$
$$= (P_1 + P_2)OE - (P_1 \times EA) + (P_2 \times EB)$$
$$= (P_1 + P_2)OE,$$

which is the moment of the resultant of P_1 and P_2 about O.

In the Fig. 89-*B*, in the same way it can be proved that the sum of the moments of the components is equal to the moment of the resultant about O.

That is, if a system of three parallel forces is in equilibrium the algebraic sum of their moments about any point in their plane of action or about any axis perpendicular to that plane is zero.

168. All these different treatments with two parallel forces can be extended to any number of parallel forces just in the same way as was mentioned in case of concurrent forces. Therefore, it can be said that if a system of parallel forces are in equilibrium the sum of their moments about any point in their plane of action or any axis perpendicular to the plane is zero. But the reverse is not always true, *i.e.*, if the sum of moments of several parallel forces be zero, the system may or may not be in equilibrium. When a number of parallel forces act on different points, though they may not be in equilibrium, it is evident that the sum of their moments about a point on the line of action of their resultant is zero, because the moment of the resultant about any point on its line of action is zero. Therefore, in the reverse case moments must be found out about any three points not in a straight line and their respective sums should be checked.

169. Couples. If in the case of two unlike parallel forces (Fig. 88) the forces be equal in magnitudes then the point O is non-existent, because from the diagram it is evident that the construction fails—the two resultants become parallel. In such cases the two forces cannot be replaced by a single force, *i.e.*, cannot have a single force as resultant. Hence, they cannot be neutralised by a single force, *i.e.*, cannot have a single equilibrant. Such two forces are said to form

a *couple.* They tend to rotate the body on which they act. The axis of rotation is the axis about which a body being acted upon by the couple, is constrained to rotate. *In Fig. 90* two forces P and P form a couple. The perpendicular distance AB, between the two forces is called the *arm* of the couple.

170. Moment of a Couple. Take (Fig. 90) any point O (O may also be taken as trace of an axis perpendicular to the plane of the

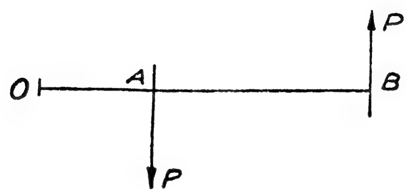


FIG. 90

couple) in the plane of the couple. Draw a straight line at right angles to the line of action of the forces cutting them at A and B respectively.

Now, the sum of the moments or turning tendency of the forces about any point O in their plane of action is equal to $P \times OB - P \times OA = P\ (OB - OA) = P \times AB$. As the point O has been chosen anywhere on the plane of action of the forces, this turning tendency is independent of the position of the point O. It only depends on the magnitude of the force and the arm of the couple. If the tendency to turning be clockwise, the couple is called *clockwise couple* and when the tendency is just the reverse, the couple is called *anti-clockwise couple.* Whether the clockwise moment is to be taken in the positive sense or the anti-clockwise is immaterial. In this book clockwise moments will be taken in the positive sense.

171. Equivalent Couples and the condition of Equilibrium. If two couples act in the same plane and if they have equal turning tendency in the same direction then they are said to be *Equivalent Couples*, *i.e.*, couples that can be interchanged one for the other without creating any change in the effect. The position in space of an equivalent couple may be anywhere in the plane. Now, if a pair of couples whose turning tendencies or moments are equal but opposite

in sense acts then the two couples will balance each other. This is proved in the following way :

Let two forces P_1, P_1 form a clockwise couple and P_2, P_2 form an anti-clockwise couple and let their moments be equal in magnitude (Fig. 91). Let their lines of action cut at A, B, C and D as shown

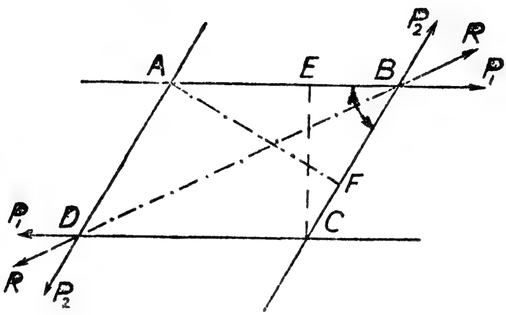


FIG. 91

in the diagram. Drop perpendiculars AF on BC and CE on AB. Then, the moments of the two couples being equal in magnitude,

$$P_1 \times CE = P_2 \times AF$$

or, $$P_1 \times CB \sin ABC = P_2 \times AB \sin ABC,$$

or, $$P_1 \times CB = P_2 \times AB,$$

or, $$\frac{P_1}{P_2} = \frac{AB}{CB}$$

Hence, AB and CB may represent, as vectors, the forces P_1 and P_2 respectively, and because $ABCD$ is a parallelogram the resultant of P_1 and P_2 at B will act along the diagonal DB and in the direction D to B. Similarly, the forces P_1 and P_2 at D have an equal amount of resultant but opposite in direction, *i.e.*, B to D. Thus, the two resultants being equal and opposite in direction will neutralise each other. Therefore, the two couples having turning moments of equal magnitudes but opposite in direction balance each other.

172. Resultant of Couples in the Same Plane. Let three couples act in the same plane (Fig. 92-*a*) whose moments are $-P_1a_1$, $+P_2a_2$ and $+P_3a_3$ respectively. For each of these three couples an equivalent couple can be determined in the same plane having a common

arm equal to 'a' and the equivalent couples are placed in such a way that the lines of action of the forces coincide as shown in Fig. 92-*b*. The equivalent couples are,—P'_1a, P'_2a and P'_3a respectively. Now, adding forces on each side of the arm two equal and opposite forces

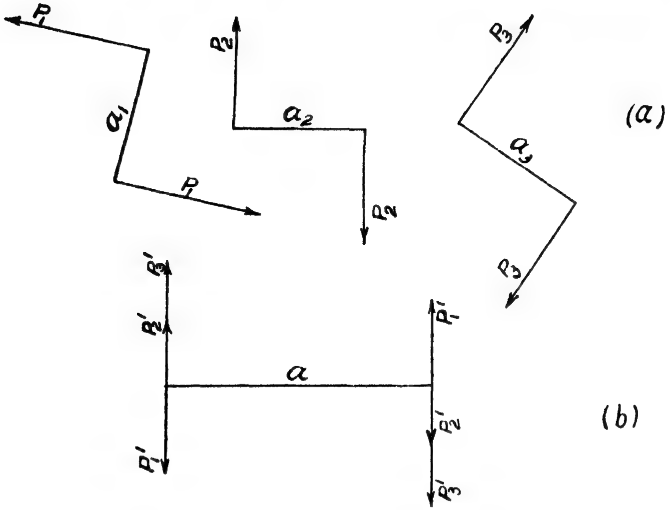


FIG. 92

of amount, $P'_3 + P'_2 - P'_1$ are found to act, which form a resultant couple producing a moment, equal to $(P'_3 + P'_2 - P'_1)\ a$. Thus,

$$(P'_3 + P'_2 - P'_1)\ a = P'_3a + P'_2a - P'_1a$$
$$= P_3a_3 + P_2a_2 - P_1a_1.$$

173. Resultant of a Couple and a single Force. Let P_1a_1 be a couple and P be the single force (Fig. 93-*a*). Choose an arm a for an equivalent couple for P_1a_1 at right angles to the line of action of the force P such that $P_1a_1 = P\,a$. Place the equivalent couple and the force as shown in the diagram (Fig. 93-*b*). It is evident from the diagram that the resultant of the system is a single force P, which lies to the left of the given force P at a distance of 'a' from it and parallel and in the same direction to it. If the couple would have been in the opposite direction then the resultant would act to the right of the given force.

It is to be marked here that the position of the resultant in space is such that its moment about any point in the line of action of the given force is of the same sign with the moment of the given couple.

Hence, from the above result it is also evident that a single force can be resolved into an equal and parallel force and a couple

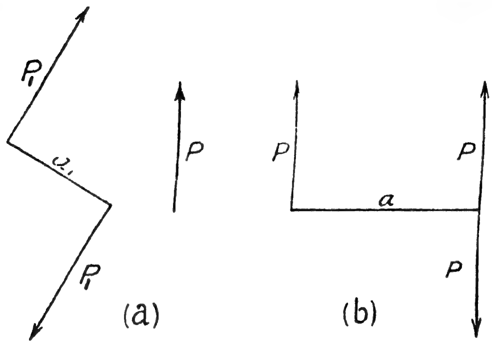


FIG. 93

the moment of which may be anything depending on the arm chosen but the magnitudes of the forces forming the couple must be equal to the given force. The direction of the couple may be anything if it be not mentioned.

174. Resultant of several co-planer Parallel Forces. Let five forces P_1, P_2, P_3, P_4 and P_5 act on five different particles (Fig. 94). Their resultant or the equilibrant and its point of application can be found out, as shown in the diagram, taking two forces

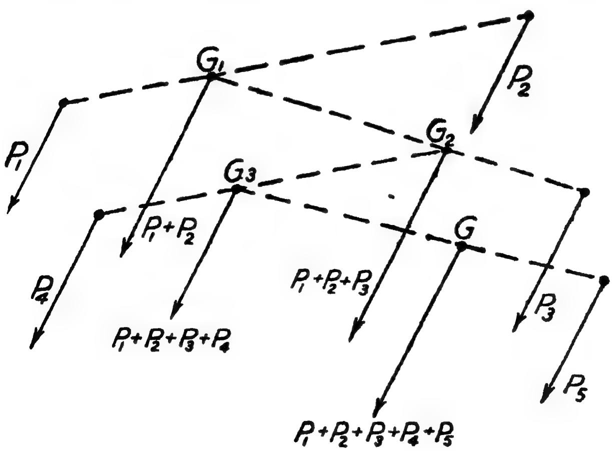


FIG. 94

at a time (Art. 164). The resultant of P_1 and P_2 is $P_1 + P_2$ acting at a point G_1 such that the point G_1 divides the line joining the points of application of P_1 and P_2 in the inverse ratio of the two forces. Similarly, this resultant with the force P_3 will produce a resultant $(P_1 + P_2 + P_3)$ to act at G_2, and so on. Thus, the resultant of the five forces, the magnitude of which is equal to $P_1 + P_2 + P_3 + P_4 + P_5$, is found to act at the centre G of the five parallel forces by the gradual method. In this way the resultant or the equilibrant and its point of application of any number of forces can be found out. The magnitude of the resultant is obviously the algebraic sum of the magnitudes of the component forces ; and because the directions of the forces are parallel the same result will be obtained from the geometric sum of the forces.

175. Now, if the points of application of the forces remaining unchanged the direction is changed by the same amount in the same sense, then it is clear that the point of application of the resultant force and hence the equilibrant, *i.e.*, the centre of the system of the parallel forces will remain the same. Again, the points of application remaining the same, if the magnitudes of the forces be proportionally altered, whether the direction remains the same or is changed by the same amount in the same sense, the centre of the system of the parallel forces will not be changed. Thus, the centre of a system of parallel forces occupies the same position in space, if

I. The points of application of the forces remain unchanged,

II. The relative magnitudes of the forces remain the same,

III. The directions of the forces are in the same sense always.

176. Centre of a System of Parallel Forces with reference to any straight line. It is evident that to locate the actual position of a point in a plane, the point is required to be referred to two known straight lines or axes in the plane. The lines of action of the forces in a plane and their points of application are always referred to two rectangular co-ordinate axes, OX and OY. Let the co-ordinates of the points of application of five parallel forces in a system in a plane be, (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) and (x_5, y_5) respectively (Fig. 95). Then, assuming the forces to be parallel to Y-axis, by the principle of moments,

$(P_1 + P_2 + P_3 + P_4 + P_5)\ \bar{x} = P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 + P_5x_5$, where $\bar{x}$ represents the distance of the centre from the Y-axis. Again, if the directions be assumed to be parallel to the X-axis then, $(P_1 + P_2 + P_3 + P_4 + P_5)\ \bar{y} = P_1y_1 + P_2y_2 + P_3y_3 + P_4y_4 + P_5y_5$, where $\bar{y}$ represents the distance of the centre from the X-axis. Here, the co-ordinates of the centre of the five parallel forces (*i.e.*, for a

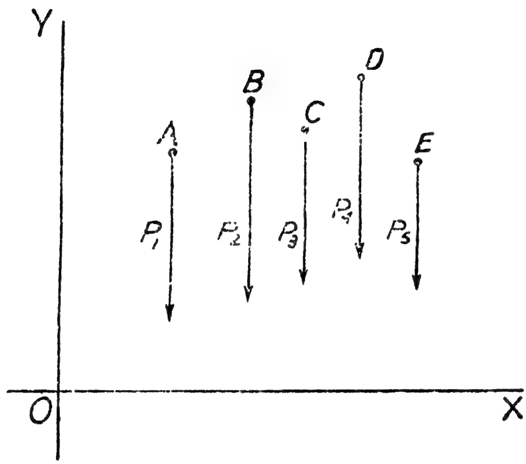


FIG. 95

system of parallel forces in general) are $\bar{x}$ and $\bar{y}$ and can be put in the forms of the following equations,

$$\bar{x} = \frac{\Sigma Px}{\Sigma P} \qquad \text{Eq. 88}$$

and $$\bar{y} = \frac{\Sigma Py}{\Sigma P} \qquad \text{Eq. 89}$$

where ΣPx and ΣPy represent the sums of the moments of the individual forces about the two axes respectively, and ΣP is the sum of the forces.

177. Centre of a System of Parallel Forces with reference to a plane.

I. *A system of two parallel forces.*

The distance of a point from a plane means its perpendicular distance from the plane.

Let the two forces P_1 and P_2 act at points A and B respectively (Fig. 96). It is required to find out the distance of the centre of these two parallel forces from, say, Y-plane The position of the centre, G, will be such that, $\frac{GB}{AB} = \frac{P_1}{P_1 + P_2}$ (*Eq.* 86)

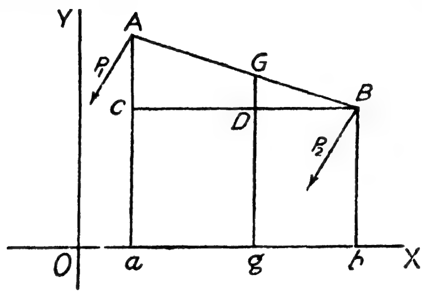


FIG 96

Drop perpendiculars Aa, Gg and Bb from A, G and B respectively on the Y-plane Draw a straight line BC, parallel to the plane, cutting Gg and Aa at D and C respectively Then, $Gg = GD + Dg$ Now, from similarity of triangles, $\frac{DG}{CA} = \frac{GB}{AB} = \frac{P_1}{P_1 + P_2}$. Therefore,

$DG = AC. \frac{P_1}{P_1 + P_2}$ and hence, $Gg = Dg + AC. \frac{P_1}{P_1 + P_2}$.

But, $Dg = Ca = Bb$

Let the perpendicular distances of A and B from the Y-plane be measured as y_1 and y_2 respectively, and that of G be $\bar{y}$. Then,

$$\bar{y} = y_2 + \frac{P_1}{P_1 + P_2}(y_1 - y_2) = \frac{P_1 y_1 + P_2 y_2}{P_1 + P_2} \qquad \textit{Eq. } 90$$

Similarly, if x_1 and x_2 be the perpendicular distances of A and B from the X-plane, then, the distance of the centre from the X-plane,

$$\bar{x} = \frac{P_1 x_1 + P_2 x_2}{P_1 + P_2} \qquad \textit{Eq. } 91$$

II. *A system of parallel forces (in one plane).*

The distance of the centre of a system of parallel forces, P_1, P_2, P_3, P_4, etc. acting at different points in a plane can be determined by extending the method of considering two forces at a time, as in the previous case.

Let $\bar{x}$ and $\bar{y}$ be the required distances of the centre from the X and Y planes respectively, and let the distances of the points of application of the forces from the two planes be, (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), etc. respectively. Then,

$$\bar{x} = \frac{P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 + \ldots\ldots}{P_1 + P_2 + P_3 + P_4 + \ldots\ldots\ldots}$$

$$= \frac{\Sigma Px}{\Sigma P} \qquad \ldots\ldots\ldots\ldots Eq.\ 92$$

Similarly, $$\bar{y} = \frac{\Sigma Py}{\Sigma P} \qquad \ldots\ldots\ldots Eq.\ 93$$

From the above two examples it is also evident that the result is quite independent of the positions of the points of application of the forces in space till their perpendicular distances from the planes remain the same. It is to be marked that in the above case, *i.e.*, in case where the lines of action of the forces be in different planes, the reference of a third plane is required to locate the actual position in space of the centre of a system of parallel forces. Let that plane be the Z-plane. The distance of the centre from the Z-plane is, in the same way, $\bar{z} = \frac{\Sigma Pz}{\Sigma P}$. Thus, the distances of point from the three planes, *i.e.*, the position in space of the point is represented by

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}, \quad \bar{y} = \frac{\Sigma Py}{\Sigma P}, \quad \bar{z} = \frac{\Sigma Pz}{\Sigma P} \qquad \ldots\ldots\ldots\ldots Eq.\ 94$$

The terms,

P_1x_1, P_2x_2, P_3x_3, P_4x_4, P_5x_5 ....

P_1y_1, P_2y_2, P_3y_3, P_4y_4, P_5y_5 ....

and P_1z_1, P_2z_2, P_3z_3, P_4z_4, P_5z_5 ....

etc.

are called the *plane moments* of the forces with respect to the planes referred to.

178. From the discussion of the cases of co-planer parallel forces it is clear that a system of parallel forces may be

(*i*) replaced by a single force (*i.e.*, by their resultant),

(*ii*) found to be in equilibrium (*i.e.*, the resultant is nil),

or, (*iii*) reduced to a single couple.

179. Conditions of Equilibrium of a system of co-planer Parallel Forces.

I. Vector diagram will be a closed figure, *i.e.*, the geometric sum of the forces is zero. In drawing the vector diagram the vectors will lie on the same straight line.

II. Algebraic sum of the forces must be equal to zero, *i.e.*, $\Sigma F = 0$. If the lines of action of the forces be not parallel to either of the two co-ordinate axes, the forces can be resolved along the two directions of the axes, which are generally the vertical and the horizontal, and $\Sigma V = 0$ and $\Sigma H = 0$, *i.e.*, $\Sigma Y = 0$ and $\Sigma X = 0$.

III. Algebraic sum of the moments of the forces about any point in the plane or about any axis at right angles to the plane of action of the forces must be equal to zero, *i.e.*, $\Sigma M = 0$. It is to be remembered that the point or the axis must be away from the line of action of the resultant.

It is to be marked here that unlike the concurrent forces though the first and the second conditions may be satisfied in a system of parallel forces, the third condition may not be fulfilled, which is evident from the case of a couple. Therefore, in order to maintain equilibrium the system should satisfy the third condition also. Again, it may be noticed that if the third condition is found to be satisfied then the first and the second conditions will automatically be satisfied ; because the third condition being fulfilled there can have no resultant, and therefore, the other two conditions are satisfied. The third condition being fulfilled it is clear that there can have neither the motion of translation, nor the motion of rotation.

Only two unknown elements of this force system can be determined with the help of the conditions stated above.

Illus. Ex. 92. *A heavy uniform rod rests on two pegs, in the same horizontal line, one foot apart If the pressure on the pegs are in the ratio 1 2, find the distance of the pegs from the middle point of the rod*

Let P_1 and P_2 be the forces on the pegs at A and B respectively (Fig 97) of which P_1 is bigger A system of three parallel forces are acting on the rod—P_1, P_2 and the weight of the rod, which act through E, the middle point of the rod The point E will divide the length AB in such a way

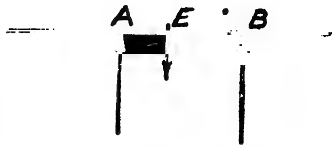


FIG 97

that $\dfrac{P_1}{P_2} = \dfrac{BE}{AE}$. But, $\dfrac{P_1}{P_2} = \dfrac{2}{1}$

$$\therefore \frac{BE}{AE} = 2, \text{ or, } BE = 2\ AE$$

Again, $BE + AE = 12$ inches

Hence, $AE = 4$ inches, and $BE = 8$ inches

Illus. Ex. 93. *A uniform beam 14 feet long and weighing 120 lbs, rests in a horizontal position on two props distant 5 feet and 3 feet from the two ends If each prop can support a pressure equal to a weight of 200 lbs. without giving way, find the greatest magnitude of equal weights that can be suspended from the two ends of the rod*

Taking R_B (Fig 98) $= 200$ lbs, the sum of the moments of the forces about $A = W \times 9 - 200 \times 6 + 120 \times 2 - W \times 5 = 0$, where W is the

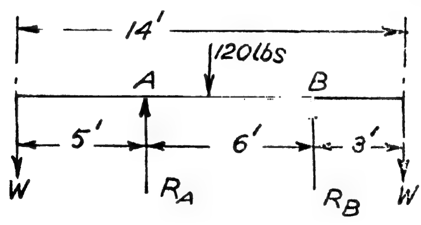


FIG 98

weight suspended and R_A & R_B are the reactions at A and B, the two props, respectively

From the solution, $W = 240$ lbs Then, $R_A = 240 + 240 + 120 - 200$
$= 400$ lbs, which is impossible

Again, taking $R_A = 200$ lbs, the sum of the moments about B

$$= W \times 3 - 120 \times 4 + 200 \times 6 - W \times 11 = 0$$

or, $W = 90$ lbs, which is possible

Because, R_B then becomes $= 90 + 90 + 120 - 200 = 100$ lbs.

Hence, under the conditions the weights suspended must be equal to 90 lbs. each.

Illus. Ex. 94. *A rod without weight is acted upon by like parallel forces 10, 2, and 5 at distances 2, 4 and 6 from one extremity. If the rod is of length 8 ft., find the magnitude and the line of action of the resultant.*

The positions of the forces will be as shown in Fig. 99. The magnitude of the resultant, of course, will be the sum of the forces as they are parallel

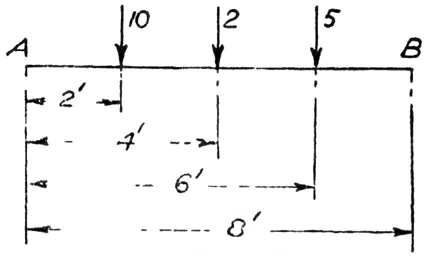


FIG. 99

and in the same direction, Therefore, it is equal to $(10 + 2 + 5) = 17$, in the same unit with the component forces measured and also the direction will be the same with the component forces. Now, taking the moments about A,

$$(10 \times 2) + (2 \times 4) + (5 \times 6) = 17 \times x,$$

where x is the distance of the line of action of the resultant from A.

$$\text{Therefore, } x = \frac{58}{17} = 3\frac{7}{17} \text{ feet.}$$

180. Co-planer Non-concurrent Forces in general. When the lines of action of all the forces in a system do not meet at a point the forces are said to be non-concurrent. Some of these forces may be intersecting while some other may not. In such cases the most ordinary method of compounding is to consider two forces at a time. The resultant of any two of the forces is added with a third force. Next, with the resultant of this pair, a fourth force is added, and thus, one by one, all the forces are added and the final resultant is obtained. The equilibrant is just equal and opposite to the resultant.

The second method is to draw the vector diagram and it is evident that to keep a body in equilibrium the vector diagram will be a closed figure.

Thirdly, the method of resolution is adopted. The resultant, $R = \sqrt{(\Sigma V)^2 + (\Sigma H)^2}$.

In all the above methods only the magnitude and direction of the resultant or the equilibrant are possible to be found out, but not the position in space. The position in space is determined by the principles of moments. The algebraic sum of the moments of the forces is equal to the moment of the resultant and thus the distance of the line of action of the resultant from the point about which the moments are taken is determined.

There is a fourth method which is as follows :

We know that each force can be resolved into a single force and a couple. With the help of this, the resultant of any system of non-concurrent co-planer forces can be found out. Let P_1, P_2, P_3 and P_4 be the four non-concurrent forces acting as shown in Fig. 100.

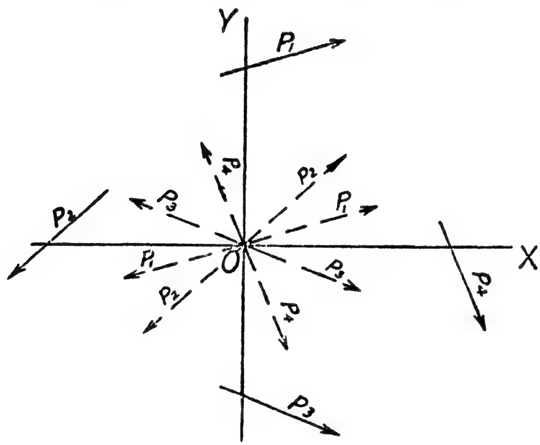


FIG. 100

Through any point O in the plane draw two rectangular co-ordinate axes OX and OY. Resolve each of the forces into a single force and a couple by introducing equal and opposite forces at the origin as is done in the diagram. Thus four sets of couple and four concurrent forces are obtained. Now it is easy to compound the four couples and the four concurrent forces separately and in doing so one resultant couple and a resultant force are obtained. Let the resultant moment due to the couples be M and let R be the resultant of the concurrent forces. Now, the resultant of these, M and R, will be a single force of magnitude R acting at a distance, $\frac{M}{R}$, so that it produces a moment about O in the same sense with M.

181. In this case too it is clear that the system of forces can produce,

(*i*) a single resultant force,

or, (*ii*) a zero resultant, *i.e.*, may be in equilibrium,

or, (*iii*) a single couple (the resultant of the concurrent vectors being zero).

182. Conditions of Equilibrium:—

1. Vector diagram will be a closed figure, *i.e.*, Geometric sum is equal to zero.
2. $\Sigma H = 0$ & $\Sigma V = 0$, *i.e.*, $\Sigma X = 0$ & $\Sigma Y = 0$.
3. $\Sigma M = 0$.

The conditions are just the same in all the systems of co-planer balanced forces. These conditions may be taken as the general conditions for all systems of co-planer forces. It should be always remembered that in cases of concurrent forces if any of the conditions is fulfilled the other two are automatically fulfilled, but in the other two systems either the first or the second condition must be fulfilled and the third condition also must be satisfied.

It is evident that the maximum number of unknown elements of a balanced system of co-planer forces that can be determined with the help of the conditions stated above is *three*.

183. Thus,

1. In a balanced system of co-planer concurrent forces the number of unknown elements that can be determined is TWO.
2. In a balanced system of co-planer parallel forces the number is TWO.
3. In a balanced system of co-planer forces in general the number is THREE.

Illus. Ex. 95. *Find the resultant of a system of co-planer forces as shown in Fig. 101. The forces are all in pounds.*

$$\Sigma H = 30 + 40 \cos 45 - 50 \cos 30 - 20 + 25 \cos 45 - 10 \sin 30$$
$$= 7.6615 \text{ lbs.}$$

$$\Sigma V = 40 \sin 45 + 15 + 50 \sin 30 - 25 \sin 45 - 10 \cos 30$$
$$= 41.9465 \text{ lbs.}$$

$$\therefore \quad R = \sqrt{(7.6615)^2 + (41.9465)^2} = 42.5 \text{ lbs.}$$

Again,

$$\Sigma M = 40 \times 5 + 50 \times 4 + 25 \times 5 - 10 \times 3 = 495 \text{ lb. ft.}$$

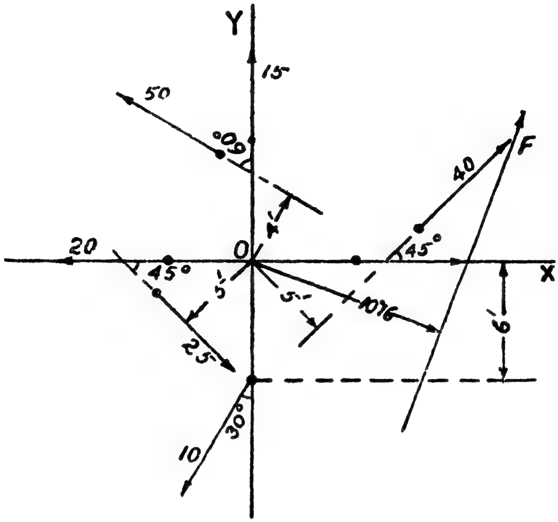


FIG. 101

Therefore, the perpendicular distance of the resultant from the point $O = \dfrac{495}{42.5} = 10.165$ feet.

If the angle made by the line of action of the resultant with the X-axis be α, then, $\tan \alpha = \dfrac{41.9465}{7.6615} = 5.46$. Therefore, $\alpha = 79.8°$.

Illus. Ex. 96. *A triangular roof-frame ACB has a span AB = 30 ft. The member AC = 20 ft. and the height of C from AB = 10 ft. The loads are applied as shown in Fig. 102 at right angles to AC and a vertical load at the middle of BC. If the end of the frame at A is fixed and that at B is free resting on a smooth surface, find the reactions at the ends.*

Under the condition given the reaction at B is vertical. Now, the distance of the load on BC from the point A is 23.65 feet.

$$\sqrt{20^2 - 10^2} = 17.3. \qquad 17.3 + \tfrac{1}{2}(30 - 17.3) = 23.65 \text{ feet.}$$

Taking the moments of the forces about the point A,

$$2 \times 10 + 1 \times 20 + 1 \times 23.65 - R_B \times 30 = 0$$

$\therefore \quad R_B = 2.121\ T$, where R_B is the reaction at B.

Again, $V_A = 1 \sin 60 + 2 \sin 60 + 1 \sin 60 + 1 - 2.121 = +2.343\ T.$

$$H_A = -(1 \cos 60 + 2 \cos 60 + 1 \cos 60) = -2\ T$$

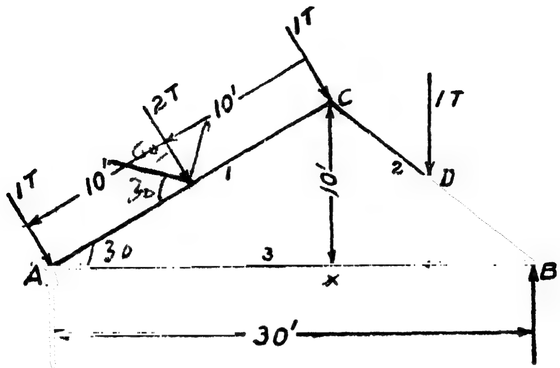


FIG. 102

Therefore, R_A, the reaction at $A = \sqrt{(2.343)^2 + (2)^2} = 3.1\ T$.

From the vertical and the horizontal components of the reaction it is clear that the line of action of the reaction is in the second quadrant. If α be the angle, $\tan \alpha = \dfrac{2.34}{2} = 1.17.$

$\therefore \quad \alpha = 49.5^\circ.$

Illus. Ex. 97. *Find the resultant of three forces 1, 2, 3 acting along the sides AB, BC, CD of a square ABCD.*

The sum of the horizontal components of the forces (Fig. 103)

$$= 1 - 3 = -2.$$

The sum of the vertical components $= -2$

Therefore, the resultant $= \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}.$

The inclination of the resultant with the horizontal component will be such that,

$\tan \alpha = \dfrac{2}{2} = 1$, where α is the inclination.

$\therefore \quad \alpha = 45^\circ$. From the directions of the components it is evident that the direction of the resultant is from upwards to downwards. Now the distance of the line of action of the resultant from a definite point, say, D, is found out by the principle of moments as follows :

Taking the moments about D, $1 \times S + 2 \times S = 2\sqrt{2} \times d$, where S represents the length of the side of the square and d the required distance.

From this relation, $d = \dfrac{3\,S}{2\sqrt{2}}$.

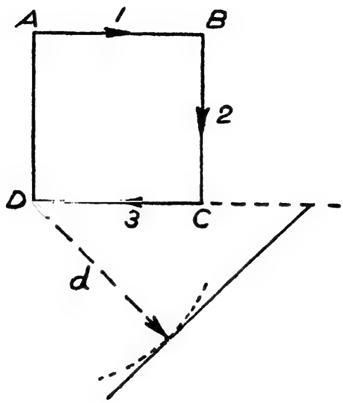


FIG. 103

Illus. Ex. 98. *Fig. 104-a shows a construction made of three members AB, CG & DE. Loads of 2000 lbs. and 1000 lbs. are applied vertically at E & G as shown. The lower end of the frame is assumed to take up the full vertical thrust. Determine the vertical and horizontal components of the forces acting at A & B, and also at the joints C, D & E.*

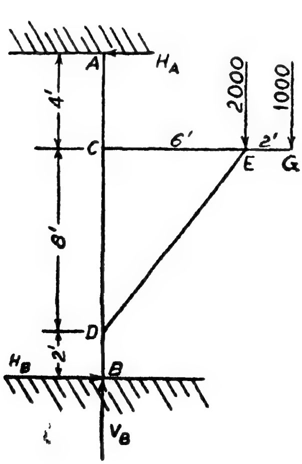


FIG. 104-a

The problem will be solved part by part as follows:

1. The whole construction will be taken as a single rigid body under the action of a balanced system of forces.

2. Three members, *CG, DE* & *AB*, will separately and singly be taken as members in equilibrium under the action of forces acting on them, *i.e.*, considering each of them as a free body.

I. Consideration of the whole frame as a rigid body. Taking moment about *B*,

$\Sigma M = 2000 \times 6 + 1000 \times 2 - H_A \times 14 = 0$ $\therefore$ $H_A = 1000$ lbs.

$\Sigma H = -1000 + H_B = 0$ $\therefore$ $H_B = 1000$ lbs.

$\Sigma V = -2000 - 1000 + V_B = 0$ $\therefore$ $V_B = 3000$ lbs.

II. Member *CG*.

CG is in equilibrium under the action of the forces as shown in Fig. 104-*b*.

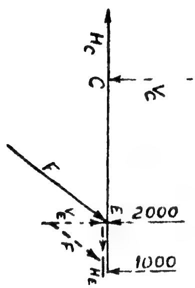


FIG. 104-*b*

$\frac{V_E}{H_E} = \frac{4}{3}$. $\therefore$ $H_E = \frac{3}{4} V_E$

Now, taking moment about *C*,

$\Sigma M = 2000 \times 6 + 1000 \times 8 - V_E \times 6 = 0$

$\therefore$ $V_E = 3333.3$ lbs.

$\therefore$ $H_E = \frac{3}{4} \times 3333.3 = 2500$ lbs.

$\Sigma H = 2500 + H_C = 0$

$\therefore$ $H_C = -2500$ lbs.

$\Sigma V = 2000 + 1000 - 3333.3 + V_C = 0$

$\therefore$ $V_C = 333.33$ lbs. (directions of the two components will be as shown).

III. Member *ED* (Fig. 104-*c*).

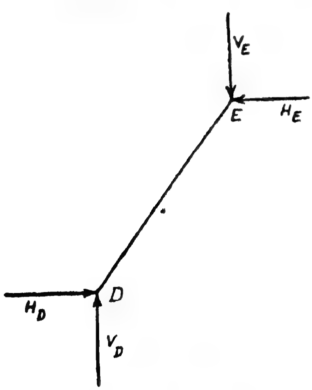


FIG. 104-*c*

$\Sigma H = -H_E + H_D = 0$

$\therefore$ $H_D = H_E = 2500$ lbs.

$\Sigma V = 0$,

i.e., $V_D = V_E = 3333.3$ lbs.

The directions are shown in the diagram.

The force acting through

$DE = \sqrt{2500^2 + 3333.3^2} = 4166$ lbs

The member is in compression.

IV. Member *AB* (Fig. 104-*d*).

From the forces acting on it, it is clear that $\Sigma H = 0$ & $\Sigma V = 0$.

It is to be noted here that the problem has been solved on the assumption that every member of the frame is a rigid body. But actually the members are elastic bodies and further treatment is necessary to deal with the internal resistance of the members that appears out of the effect of the load on them. Loads at different points create bending and its effect has been discussed in all the books on Strength of Materials.

It is to mention here that the problem can also be solved by the condition of equilibrium $\Sigma M = 0$ only. Student should try to get the results by this method and this should be taken as one of the home tasks (Problem 203).

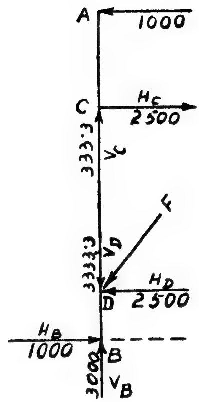


FIG. 104-*d*

METHOD OF SECTION

184. By this method the internal stresses in the members of a structure such as, roof frame, girder, etc. by which resistances are offered by different members of the construction against the actions of the loads applied, are determined. Three general assumptions are made for this method,—(1) the members are in the same plane, *i.e.*, the lines of action of the forces in the members are co-planer, (2) to imagine a plane of section at right angles to the plane of the frame to cut a number of members, and (3) to consider the portion of the frame on either side of the section as a rigid body in equilibrium under the actions of external forces acting in the portion and the internal stresses produced in the members. Problems are generally solved with the help of the principle of moments, *i.e.*, the condition, $\Sigma M = 0$, and the method of resolution, *i.e.*, the conditions, $\Sigma V = 0$ and $\Sigma H = 0$ The important point in this method is to select the position of the plane of section. It should be such that in each consideration there is only one unknown force as is shown in the following examples.

Illus. Ex. 99. *A roof-frame rests on two supports at its ends and are loaded as shown in the Fig. 105. Find the forces in the members DC, DJ and AJ and determine whether they are in tension or compression.*

The reactions, R_A and R_B, each $= (1 + 2 + 2 + 2 + 1)/2 = 4$ tons. Take a section XX to cut those members as shown. Consider the left-hand portion

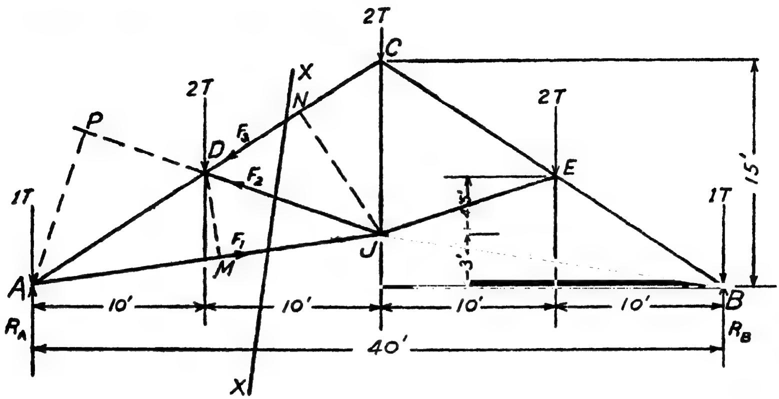


FIG. 105

of the section to be in equilibrium and take all the clockwise moments as positive and anti-clockwise moments as negative. Now,

(1) $AC = \sqrt{20^2 + 15^2} = 25$ ft. $\quad\therefore\quad AD = 12.5$ ft.

$\angle BAC = \tan^{-1} \dfrac{15}{20}$. $\quad\therefore\quad \angle BAC = 36.9°$

and $\angle BAJ = \tan^{-1} \dfrac{3}{20}$. $\quad\therefore\quad \angle BAJ = 8.5°$

$\angle JAC = 36.9 - 8.5 = 28.4°$. The perpendicular distance of AJ from D, *i.e.*, $DM = AD \sin 28.4 = 12.5 \times .4756 = 5.944$ ft.

Now, taking moment about D, $(4 - 1) \times 10 + F_1 \times 5.944 = 0$, or,

$F_1 = -\dfrac{30}{5.944} = -5.047$ tons, where F_1 is the force acting through AJ.

It is found that F_1 is producing an anti-clockwise moment because the force is negative. Therefore, the direction is towards J, *i.e.*, the member JA is pulling the joint A. When a member pulls a joint the member is in tension. Therefore, the member JA is in tension.

(2) Again, $AJ = \dfrac{3}{\sin 8.5} = 20.27$ ft. JN, the perpendicular distance of AC from the point $J = 20.27 \times .4756 = 9.64$ ft.

Now, taking moments about J, $(4 - 1) \times 20 - 2 \times 10 + F_3 \times 9.64 = 0$,

or, $F_3 = -\frac{40}{9.64} = -4.15$ tons, where F_3 is the force in DC. Here, F_3 is producing an anti-clockwise moment about J, and therefore, the direction is from C to D, *i.e.*, the force is pushing the joint D. Hence, the member is in compression. When a member pushes a joint the member is in compression.

(3) $JD = \sqrt{10^2 + 4.5^2} = 10.96$ ft. Again, $\frac{12.5}{\sin AJD} = \frac{10.96}{\sin 28.4}$,

or, $\sin AJD = \frac{12.5 \times .4756}{10.96} = .5424.$ $\therefore$ $AJD = 32.85°$.

Hence, AP, the perpendicular distance of JD from A, $= 20.27 \sin 32.85 = 10.98$ ft.

Now, taking moments about A, $2 \times 10 + F_2 \times 10.98 = 0.$

$\therefore$ $F_2 = -\frac{20}{10.98} = -1.82$ tons where, F_2 is the force acting through JD.

Here, F_2 is producing an anti-clockwise moment about A, and therefore, the direction of the force is from J to D, *i.e.*, the force is pushing the joint D. Hence, the member is in compression.

Illus. Ex. 100. *A girder is loaded as shown in Fig. 106-a. The tie-bars are inclined 60° with the horizontal members. R_A, the reaction at A is 15 tons. Find the forces acting through the members FH, FG and EG, and also whether they are in tension or compression.*

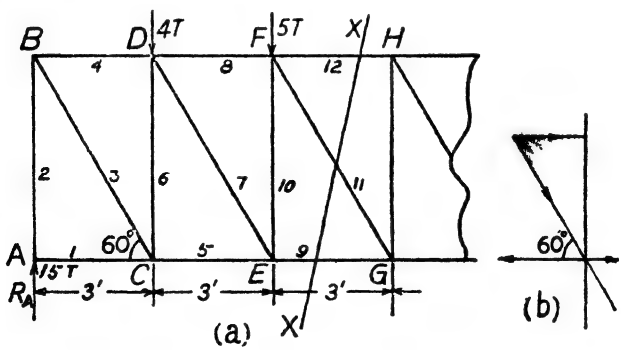


FIG. 106

Take a plane of section XX cutting the three members. Consider the left-hand portion to be in equilibrium under the action of the external loads of 4 tons and 5 tons at D and F respectively and the vertical reaction at A and also the forces acting through the three members.

The vertical members of the girder are each 3 tan 60 or 5.2 ft. long.

(1) Taking moments about G, $15 \times 9 - 4 \times 6 - 5 \times 3 +$ (force) $FH \times 5.2 = 0$. Therefore, force $FH = \frac{-135 + 24 + 15}{5.2}$

$= -18.46$ tons.

The negative sign indicates that the force FH is producing anti-clockwise moment about G, *i.e.*, the direction of the force is from H to F, *i.e.*, the force is pushing the joint F. Therefore, the member is in compression.

(2) Taking moment about F, $15 \times 6 - 4 \times 3 +$ (force) $EG \times 5.2 = 0$.

$\therefore$ (force) $EG = -15$ tons, The negative sign indicates an anti-clockwise moment about F, *i.e.*, the direction of the force is from E to G, *i.e.*, the force pulls the joint E. Therefore, the member is in tension.

(3) The vertical component of the force $FG = FG \sin 60$. Therefore, $FG \sin 60 + 15 - 4 - 5 = 0$, taking the upward vertical force as positive and downward vertical force as negative. Therefore, (force) $FG = -6.92$ tons.

Or,

The horizontal component of the force FG is $FG \cos 60$ or $FG \cos 120$, which depends on the direction of the component. Take the force towards the right as positive and towards the left as negative. Then, $FG \cos \theta = 18.46 - 15 = 3.46$ tons, where θ represents 60° or 120°. As the component is found to be positive, $FG \cos 120 = 3.46$ (Fig. 106-*b*), or $-FG \sin 30 = 3.46$, *i.e.*, $FG = -\frac{3.46}{\sin 30} = -6.92$ tons.

Now, because the vertical component has got a negative value or the horizontal component has a positive value, the direction of the force FG will be from F to G, which is clear from the vector triangle of the two components and the resultant. Therefore, the force in FG, pulling the joint F, will put the member in tension.

185. Statically Indeterminate Problems. If there are more unknown elements in a problem than can be determined on the basis of the principles of statics then the problem is statically indeterminate. It is impossible to solve the problem without assuming probable values for certain forces till the number of unknown elements is reduced to the number that can be determined from the conditions of equilibrium only.

FORCES IN DIFFERENT PLANES

186. Parallel Forces in different Planes. The method of compounding is a very simple one. It is clear that the magnitude and

direction of the resultant can be obtained by the algebraic or geometric addition. The position in space is obtained by the principle of moments. The algebraic sum of the plane moments of the forces is equal to the plane moment of the resultant.

If the forces are referred to three rectangular co-ordinate planes, X, Y and Z, and if the planes are chosen in such a way that the forces are parallel to one of the planes, say, Z-plane, and also the distances of the points of application of the forces from the three planes be represented by, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), etc. and P_1, P_2, P_3, etc. be the forces respectively, then,

$$(P_1 + P_2 + P_3 + \cdots\cdots)\,\bar{x} = P_1x_1 + P_2x_2 + P_3x_3 + \cdots\cdots$$

$$\text{or, } \bar{x} = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \cdots\cdots}{P_1 + P_2 + P_3 + \cdots\cdots} = \frac{\Sigma Px}{\Sigma P} \cdots\cdots Eq.\ 95$$

where $\bar{x}$ is the distance of the point of application of the resultant from the X-plane and ΣPx and ΣP represent the algebraic sum of the plane moments of the forces about the X-plane and the resultant force respectively. Similarly, considering the forces parallel to y and z planes respectively the distance of the resultant from the Y-plane,

$$\bar{y} = \frac{\Sigma Py}{\Sigma P} \cdots\cdots\cdots\cdots Eq.\ 96$$

and that from the Z-plane, $\bar{z} = \dfrac{\Sigma Pz}{\Sigma P}$ $\cdots\cdots\cdots\cdots Eq.\ 97$

187. Couples in Parallel Planes:—

(1) *Two couples, equal in magnitudes but opposite in direction, in two parallel planes will neutralise each other.*

The two couples may be replaced by two equivalent couples of equal arms. Let the moments of the two equivalent couples be (Fig. 107) $P_1 \times AB$ and $P_2 \times CD$. Because, $AB = CD$, $P_1 = P_2$. Join A, B, C and D. $ABCD$ will then be a parallelogram. Join AD and BC. Then, the force P_1 at A and the force P_2 at D, being equal in magnitude and in the same direction, will produce a resultant, $P_1 + P_2$ or, $2P_1$ in the same direction with them at O, the point of intersection of the diagonals. Similarly, the force P_2 at C and P_1 at B will produce a resultant, $2P_1$, at O in opposite direction to the previous

one. Thus, two equal and opposite forces act at O due to the action of the two given couples in two parallel planes. Hence, they neutralise each other.

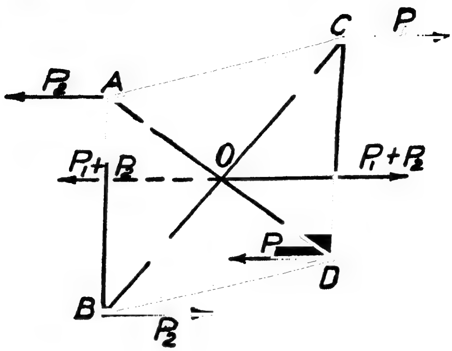


FIG. 107

(II) *The resultant of two unequal couples in two parallel planes.*

Let the moments of the couples be $P_1 \times AB$ and $P_2 \times CD$. Suppose $P_2 \times CD$ is greater than $P_1 \times AB$. Replace the bigger couple by an equivalent one whose arm $C'D' = AB$ (Fig. 108). Suppose

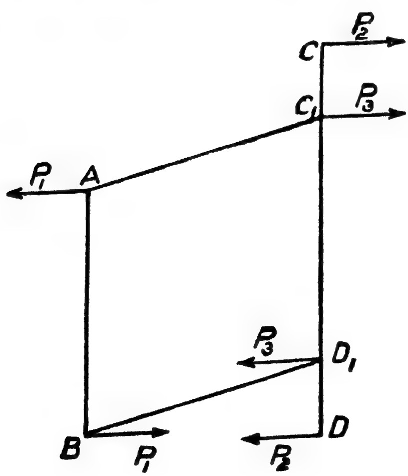


FIG. 108

the magnitude of the force, then, changes from P_2 to P_3. Then $P_3 \times C'D'$, *i.e.*, $P_3 \times AB$ must be equal to $P_1 \times AB + P_4 \times AB$,

when $P_3 = P_1 + P_4$. Thus, two equal and opposite couples in two different planes neutralise each other, and the only residue moment of the couple is $P_4 \times AB$, which is the resultant of the two unequal couples in two parallel planes. Therefore, the resultant of the two unequal couples in two different planes produces a moment equal to the algebraic sum of the moments of the two individual couples in the direction of the moment of the bigger couple and the plane of action of the resultant couple is the plane of the bigger couple.

Proceeding in the same way it can be proved that the moment of the resultant of a system of couples in different parallel planes is equal to the algebraic sum of the moments of the individual couples.

188. Couples in planes making angles with each other but each being at right angles to a common plane. A couple may have any position in space in the plane, but its moment is always constant in magnitude and direction. Therefore, it can be represented by a definite vector. The vector representing the moment of a couple may be drawn from any point in the plane of the couple and at right angles to it—its length represents the magnitude in a definite scale and the direction is indicated in the way as explained below :

The couple is studied, whether it is clockwise or anti-clockwise. The direction of the vector representing the moment of the couple and the method of drawing the vector is shown in the following example. The vector representing the moment of the couple is called the *moment axis of the couple.*

Let there be a couple as shown by the full lines in the plane $ABCD$ (Fig. 109). Also, let the plane be represented by its trace AB (or CD) on a plane at right angles to it. The vector is drawn from any point in AB and at right angles to it. If the couple is clockwise (as it is in this case) then it is drawn on the right-hand side

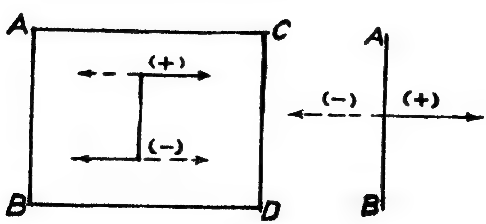


FIG. 109

of the line, AB, representing the plane and an arrow-head is put as shown. In case of an anti-clockwise couple the vector with the arrow-head is drawn on the other side as shown by the dotted line to the left of AB.

189. A vector with reference to three mutually perpendicular planes. Any vector may be resolved into components in three directions at right angles to each other. Let the vector OP represent a force P (Fig. 110). Through O draw three rectangular co-ordinate axes, OX, OY and OZ. Firstly, resolve OP into two components

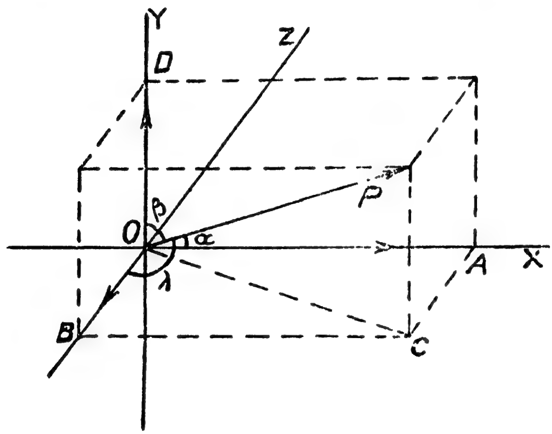


FIG 110

OD and OC, along OY and at right angles to it so that OD, OP and OC be in the same plane. Next, in the same way, resolve OC into two components OA and OB along OX and OZ respectively. Then, $OP^2 = OD^2 + OC^2 = OD^2 + OA^2 + OB^2$.

Let the angle $XOP = \alpha$, angle $YOP = \beta$ and angle $ZOP = \lambda$, then, $OA = P \cos \alpha$, $OD = P \cos \beta$ and $OB = P \cos \lambda$.

$$\text{Thus, } P = \sqrt{OA^2 + OB^2 + OD^2} \text{ and } \cos \alpha = \frac{OA}{P},$$

$$\cos \beta = \frac{OD}{P} \text{ and } \cos \lambda = \frac{OB}{P} \quad \cdots \quad Eq. 98$$

190. Compounding of Couples in different planes.

(I) *Two couples in different planes.* Let M_1 and M_2 be the

moments of two couples in two different planes, A and B respectively. The angle between the plane is θ, which is shown in the diagram (Fig. 111) by the traces, AA and BB, of the two planes respectively. From a point O common to both the planes draw two moment axes to represent M_1 and M_2 at right angles to AA and BB respectively. Then the resultant moment (which is quite clear from the diagram), $M = \sqrt{M_1^2 + M_2^2 + 2M_1 M_2 \cos \theta}$. If α be the angle between M and M_1,

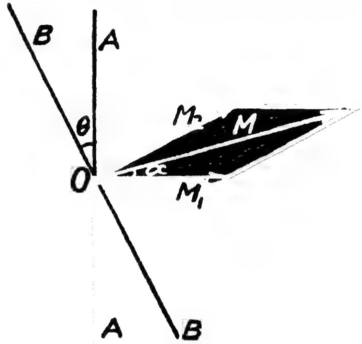


FIG. 111

Then, $\dfrac{M_2}{\sin \alpha} = \dfrac{M}{\sin (180 - \theta)}$

$= \dfrac{M}{\sin \theta}$, or, $\sin \alpha = \dfrac{M_2}{M} \sin \theta$

That is, compounding the two couples a resultant couple is obtained which acts in a plane making an angle α with the plane A and the moment of which is equal to M. Here, in this case, the sense of the resultant couple is found to be the same with that of the two component couples.

Now, if the two planes be at right angles to each other, then, θ being $90°$, $M = \sqrt{M_1^2 + M_2^2}$. If α be the angle between M and M_1,

$\tan \alpha = \dfrac{M_2}{M_1}$ and $\sin \alpha = \dfrac{M_2}{M}$.

(II) *A number of couples, each in a different plane.* Let the couples in different planes produce moments M_1, M_2, M_3, M_4, etc. Because the moment axes can be drawn from any point in the planes of the couples, all the vectors may be drawn as concurrent vectors as shown in the diagram (Fig. 112). At O, a common point, draw three rectangular co-ordinate axes, OX, OY and OZ. Then following the method of resolution as described in Art. 189, all the vectors are resolved along OX, OY and OZ respectively and the components along the three axes are added separately. Let them be equal to ΣM_x, ΣM_y and ΣM_z respectively. Then if M be the resultant moment,

$$M = \sqrt{\Sigma M_x^2 + \Sigma M_y^2 + \Sigma M_z^2} \qquad \cdots\cdots\cdots\cdots Eq.\ 99$$

If α, β and λ be the angles made by M with the axes OX, OY and OZ respectively, then,

$$\cos\alpha = \frac{\Sigma M_x}{M}, \cos\beta = \frac{\Sigma M_y}{M} \text{ and } \cos\lambda = \frac{\Sigma M_z}{M}$$

$$\cdots Eq.\ 100$$

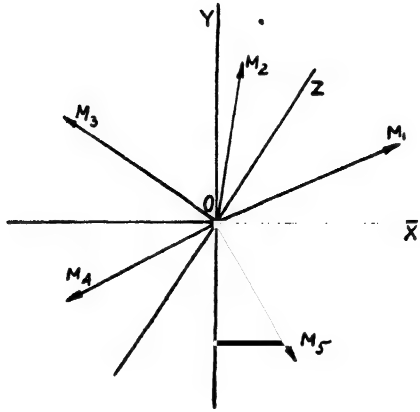


FIG. 112

191. Compounding of Forces in general in different planes.

CONCURRENT FORCES

(I) *Method of Resolution.* Through the common point of meeting three rectangular co-ordinate axes are drawn. If the forces be P_1, P_2, P_3, etc. making angles α_1, α_2, α_3 etc., β_1, β_2, β_3 etc. and λ_1, λ_2, λ_3 etc. respectively with the three axes, then their components along the three axes are $P_1 \cos\alpha_1$, $P_1 \cos\beta_1$, $P_1 \cos\lambda_1$; $P_2 \cos\alpha_2$, $P_2 \cos\beta_2$, $P_2 \cos\lambda_2$; $P_3 \cos\alpha_3$, $P_3 \cos\beta_3$, $P_3 \cos\lambda_3$, etc. respectively. Add all the components along the axes separately (adjusting proper signs according to the directions), then,

$$\Sigma X = P_1 \cos\alpha_1 + P_2 \cos\alpha_2 + P_3 \cos\alpha_3 + \cdots$$

$$\Sigma Y = P_1 \cos\beta_1 + P_2 \cos\beta_2 + P_3 \cos\beta_3 + \cdots$$

$$\Sigma Z = P_1 \cos\lambda_1 + P_2 \cos\lambda_2 + P_3 \cos\lambda_3 + \cdots\cdots$$

where, ΣX, ΣY and ΣZ represent the sum of the components along X, Y and Z axes respectively.

Now, if R be the resultant of the system, then, according to Art. 189, $R = \sqrt{\Sigma X^2 + \Sigma Y^2 + \Sigma Z^2}$.

Again, if R makes α, β and λ angles with the axes OX, OY and OZ respectively, then, the direction of R is represented by

$$\cos\alpha = \frac{\Sigma X}{R}, \cos\beta = \frac{\Sigma Y}{R}, \text{ and } \cos\lambda = \frac{\Sigma Z}{R}.$$

(II) *Vector polygon.* The resultant can also be found out by drawing the vector polygon just as was done in cases of co-planer forces. Here, the polygon will not be a regular one in a single plane, and the polygon formed can be called a *gauche* polygon.

(III) *The method of considering two forces at a time can also be adopted to get the resultant of a system of concurrent forces acting in different planes.*

192. Moment of a Force about an axis (not at right angles to each other). The force is resolved into two components, one at a parallel direction to the axis and the other in a plane at right angles to it. It is evident that the component parallel to the axis has no effect on the turning tendency about the axis of rotation. Therefore, the other component produces the turning tendency and its measure is $ab \times Od$ (Fig. 113), where ab is the vector representing the component of the force in a plane at right angles to the axis of rotation, O is the trace of the axis on the same plane and Od is the perpendicular distance of ab from O. Thus, the moment of a force about an axis not in a direction at right angles to it is the moment of the component of that force in a plane at right angles to the axis. If P be the force making an angle α with the plane, which may be taken as the surface of the paper, then the projection of the vector representing P on the plane is ab and is equal to $P \cos\alpha$. Hence, the moment of P about an axis whose trace is O, $M = P\cos\alpha \times Od$.

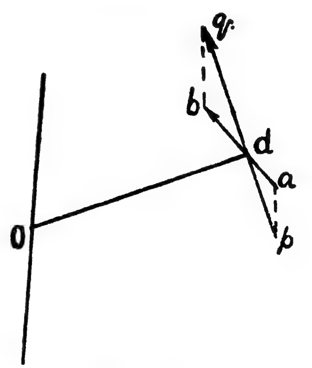


FIG. 113

193. Treatment with reference to three rectangular co-ordinate axes:—

Let the point of application of the force P have co-ordinates x, y and z and let α, β and λ be the angles made by the line of action of the force with the directions parallel to OX, OY and OZ respectively. Resolve the force along the three directions of the axes. Take the clockwise moments looking along the axes from ends X, Y and Z in the diagram (Fig. 114) towards the origin as positive and the anti-

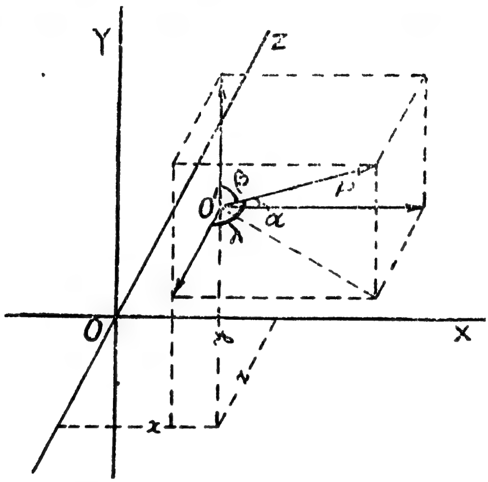


FIG. 114

clockwise ones as negative. Then, the moment of the force about an axis must be equal to the algebraic sum of the moments of the components about the axis. Hence,

M_x (moment about X-axis) $= P \cos \beta . z - P \cos \lambda . y$

M_y (moment about Y-axis) $= P \cos \lambda . x - P \cos \alpha . z$

and M_z (moment about Z-axis) $= P \cos \beta . x - P \cos \alpha . y$

............*Eq.* 101

194. Compounding of a System of Forces in different planes not meeting at a point:—

Let the vector DF represent the force P (Fig. 115), which is one of a system of non-concurrent forces acting at different planes. Let

the vector make angles α, β and λ with lines drawn through O' and parallel to the three axes as in Fig. 114. Let the co-ordinates of the

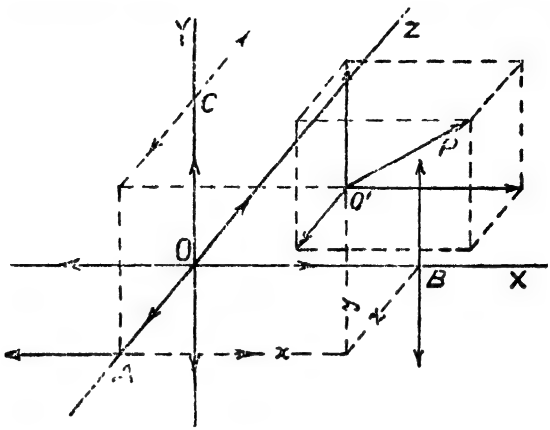


FIG. 115

point O' be x, y and z. Now, resolve the force P in three directions parallel to the axes. Let the components be $P \cos \alpha$, $P \cos \beta$ and $P \cos \lambda$. If at each of the points O and A a pair of equal and opposite forces be introduced, the magnitudes being equal to and the directions being parallel to $P \cos \alpha$, then, the effect due to the component, $P \cos \alpha$, will remain undisturbed, that is to say, the component $P \cos \alpha$ is resolved into a single force at O of equal amount to $P \cos \alpha$ and parallel and in the same direction to it and two couples equal to $P \cos \alpha . z$ and $P \cos \alpha . y$. In this way all the three components of P are resolved into three single components at O and three pairs of couples, introducing forces at A, B, C and O, just in the same way as described above.

Now, if there are several forces, P_1, P_2, P_3, etc., then, each of the forces is treated in the same way. Thus, a number of forces are obtained at the point O—the number is three times the number of forces acting—the resultant of which, a single force, can be easily found out, from the relation, $R = \sqrt{\Sigma X^2 + \Sigma Y^2 + \Sigma Z^2}$, where R is the resultant and ΣX, ΣY and ΣZ are the sums of the component forces along OX, OY and OZ respectively. Also, the same number of pair of couples is obtained from which, for each of the forces,

$$M_x = P \cos \beta . z - P \cos \lambda . y$$
$$M_y = P \cos \lambda . x - P \cos \alpha . z$$
$$\text{and } M_z = P \cos \beta . x - P \cos \alpha . y \qquad \text{(Art. 193)}$$

where, the letter P signifies the force, and α, β and λ signify the angles made by the force with the directions of OX, OY and OZ respectively, and M_x, M_y and M_z are the moments of the couples about X, Y and Z axes respectively. Now, the algebraic sum of the moments due to couples for all forces about the three axes are found out separately and let them be denoted by ΣM_x, ΣM_y and ΣM_z respectively. The resultant couple, $M = \sqrt{\Sigma M_x^2 + \Sigma M_y^2 + \Sigma M_z^2}$

(*Eq.* 98)

195. Conditions of Equilibrium.

1. R must be equal to zero, *i.e.*, the *Gauche* polygon must be a closed figure (but which is not possible to draw) and hence,

2. $\Sigma X = 0, \qquad \Sigma Y = 0, \qquad \Sigma Z = 0$

and,

3. $\Sigma M_x = 0, \qquad \Sigma M_y = 0, \qquad \Sigma M_z = 0$

are the three conditions of equilibrium for any system of non-concurrent forces acting in different planes. Now, as a general case has been treated here, it is clear that these conditions are the conditions for all the systems of forces, concurrent or non-concurrent, parallel or non-parallel, co-planer or non-co-planer. It is to be noted that in all systems of forces the vector diagram will always be a closed figure, whether it be a regular polygon or a *gauche* polygon.

It is also clear that these conditions of equilibrium for a balanced system of non-concurrent forces in different planes will enable us to determine SIX unknown elements.

Illus. Ex. 101. *Find the resultant of a system of parallel forces shown in the diagram of the Figure 116.*

$R = 150 + 50 - 200 - 100 = -100$ lbs. (downwards)

$M_x = 100 \times 5 + 50 \times 5 - 200 \times 5.5 - 150 \times 5.5 = -1175$

$M_y = -150 \times 7 - 100 \times 8 + 50 \times 4 + 200 \times 10 = +350$

(clockwise from the positive end of the axes is taken as positive)

Therefore, the co-ordinates of the point of application of the resultant, R, will be such that,

$$x = \frac{350}{100} = 3.5 \text{ feet.} \quad \text{(as moment about } Y\text{-axis is positive)}$$

$$y = \frac{-1175}{100} = -11.75 \text{ feet.} \quad \text{(moment about } X\text{-axis is negative).}$$

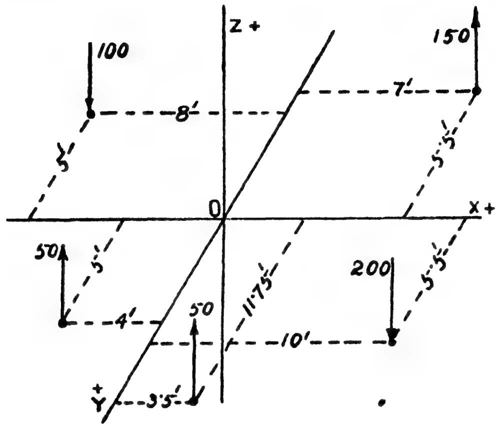


FIG. 116

Illus. Ex. 102. *A system of forces as shown in Fig. 117 (a) is in action. The line of action of the force of 50 lbs. is in the Y-plane and makes an*

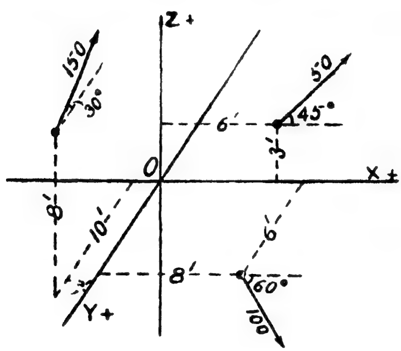


FIG. 117(*a*)

angle of 45° with the Y-plane. The line of action of the force of 150 lbs. is in a plane parallel to the X-plane and makes an angle of 30° with Z-plane.

The line of action of the force of 100 lbs. is in the Z-plane and makes an angle of 60° with the Y-plane. The distances of the points of application are given in the figure. Determine the resultant of the system.

Resolving the forces along the three rectangular co-ordinate axes,

$$\Sigma X = 50 \cos 45 + 100 \cos 60 = 85.35 \text{ lbs.}$$

$$\Sigma Y = 100 \sin 60 - 150 \cos 30 = -43.3 \text{ lbs.}$$

$$\Sigma Z = 50 \sin 45 + 150 \sin 30 = 110.35 \text{ lbs.}$$

$$\therefore \quad R = \sqrt{(85.35)^2 + (43.3)^2 + (110.35)^2} = 146 \text{ lbs.}$$

If α be the angle between R and Z, $\cos \alpha = \dfrac{110.35}{146} = .7553$

$$\therefore \quad \alpha = 40.9°$$

Again, $\sqrt{(85.35)^2 + (43.3)^2} = 95.7$. If β be the angle made by the plane containing R and Z axes, $\sin \beta = \dfrac{43.3}{95.7} = .452$, whence, $\beta = 26.9°$. OA represents the intersecting line of the Z-plane with the plane of R and OZ.

Computing the moments of the components of the forces about the three axes respectively and adding,

$$\Sigma M_x = 150 \cos 30 \times 8 + 150 \sin 30 \times 10 = 1790 \text{ lb. ft.}$$

$$\Sigma M_y = 150 \sin 30 \times 2 - 50 \sin 45 \times 6 + 50 \sin 45 \times 3$$
$$= 44 \text{ lb. ft.}$$

$$\Sigma M_z = 150 \cos 30 \times 2 + 100 \sin 60 \times 8 - 100 \cos 60 \times 6$$
$$= 653 \text{ lb. ft.}$$

$$\therefore \quad M = \sqrt{1790^2 + 44^2 + 653^2} = 1906 \text{ lb ft.}$$

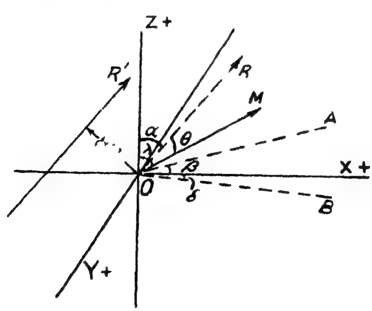


FIG. 17(b)

If λ be the angle between M and OZ, $\cos\lambda = \dfrac{653}{1906} = .3426$,

whence, $\lambda = 70°$

But, $\sqrt{1790^2 + 44^2} = 1790.7$. $\therefore$ $\sin\delta = \dfrac{44}{1790\ 7} = .02458$, where δ is the angle made by the plane containing M and OZ with OX. OB represents the line of intersection of the Z-plane with the plane of M and OZ. From the relation, $\delta = 1.4°$.

Therefore, the angle $AOB = \beta + \delta = 26.9 + 1.4 = 28.3°$, where the angle AOB is the angle between the two planes.

From Spherical Trigonometry, $\cos\theta = \cos\alpha \cos\lambda + \sin\alpha \sin\lambda \cos AOB$, where θ is the angle between M and R. Now, substituting the values, $\cos\theta = .8068$, whence, $\theta = 36.2°$.

As the moment axis M does neither coincide with R, nor is at right angles to it, resolve M into two directions along the direction of R and at right angles to it. Let M_1 and M_2 be the two components.

Then, $M_1 = M\cos\theta = 1906 \times .7221 = 1538$ lb. ft.

and $M_2 = M\sin\theta = 1906 \times .6921 = 1126$ lb. ft.

Now, combining M_2 with R, we get the resultant force R', which is equal to and in the same direction with R. Its position in space will be such that its distance is s from O in a direction perpendicular to the plane containing R and M. The value of $s = \dfrac{M_2}{R'} = \dfrac{1126}{146} = 7.7$ ft. Thus, the resultant of the system is a single force R' and a couple M_1. $R' = 146$ lbs. and $M_1 = 1538$ lb. ft. The plane of the couple is perpendicular to R' tending to create a right-handed turning when looking along R' in the opposite direction.

196. Centre of Stress. Load is the external force applied on a piece of material or construction. Owing to the application of load, internal resistive force appears in the material due to its property to overcome the action of the load. This resistive force is induced on the cross-section of the member on which the load is applied and is called *stress*. Though the total resistive force is the stress but it is always represented by the resistive force per unit area at different points of the cross-section, which is called the intensity of stress, stress intensity or unit stress at those points.

The intensity may be uniform as well as varying. In case of varying intensity, again, the variation may be uniform as well as non-uniform.

As the induced forces are, generally, parallel in direction, the principles of parallel forces are used to determine the centre of the

stress. If P be the total stress,, then, the general equation for P is $P = \int p.\, dA$, where p represents the unit stress and dA is a differential area. If the cross-sectional area is referred to the rectangular co-ordinate axes X and Y, and the intensity of stress is represented along the Z-axis, the centre of the stress will be such that,

$$\bar{x} = \frac{\int p.x.dA}{\int p.dA} \quad \text{and} \quad \bar{y} = \frac{\int p.y.dA}{\int p.dA} \quad \cdots\cdots\cdot \quad Eq.\ 102$$

where $\bar{x}$ and $\bar{y}$ are the co-ordinates of the position of the centre.

Illus. Ex. 103. *In the rectangular section given in Fig. 118-a the intensity of stress varies uniformly from O at one end to the maximum 1000 lbs. per square inch at the other, as shown in Fig. 117-b. Determine the resultant stress and locate the centre of stress with respect to the X and Y axes.*

The intensity of stress at a distance x from the Y-axis, $w = \frac{1000}{12} x$ lbs. per sq. inch.

Therefore, the resultant $R = W$ (total load) $= \int w.dA$

$$dA = 6.\, dx$$

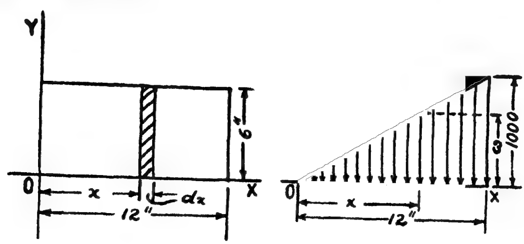
Hence, $R = \int_0^{12} \frac{1000}{12} x.\, 6.\, dx = 500 \int_0^{12} x\, dx$

Again, if x and y be the distances of the centre from the Y and X axes respectively,

$$\bar{x} = \frac{\int w.x.dA}{\int w.dA} = \frac{\int_0^{12} \frac{1000}{12} x^2 .\, 6.\, dx.}{\int_0^{12} \frac{1000}{12} x\ 6.\, dx}$$

$$= 8 \text{ inches,}$$

and, $\bar{y} = 3$ inches, because the centre of stress on each strip will be at the middle of the length of the strip as the stress intensity is uniform throughout the strip area.



(a) FIG. 118 (b)

197. *It is to be marked that in statics when the discussion of non-concurrent forces begins the idea of a rigid body arises in mind, because when the question of points of application appears we cannot treat the cases as those of particles. Bodies composing of particles, i.e., systems of particles are required, where the different particles will be acting as points of application of force.*

PROBLEMS

188. A block of stone is suspended with the help of three ropes, all in one plane. They are inclined at angles of 40°, 120° and 160° to the horizontal line counting anti-clockwise. The tensions in the first two ropes are 150 and 120 lbs. respectively. Find the pull in the third rope and also the weight of the stone. *Ans.* 58.5 lbs., ; 220 lbs.

189. In a horizontal plane five equally spaced ropes keep a pole standing in equilibrium. If the tensions in the three consecutive ropes are 2000, 2800 and 2400 lbs. respectively, find the tensions in the other two.

Ans. 2250 lbs. ; 2890 lbs.

190. In a circle draw two diameters *AB* and *CD*. If two forces acting at a particle can be represented in magnitude and direction by *AB* and *CD* respectively, prove that the third force to keep the particle at rest will be equal to 2 *BC*.

191. Find a point within a quadrilateral such that if it be acted upon by forces represented by the lines joining it to the angular points of the quadrilateral it will be in equilibrium.

Try to understand the solution from the Fig. 119.

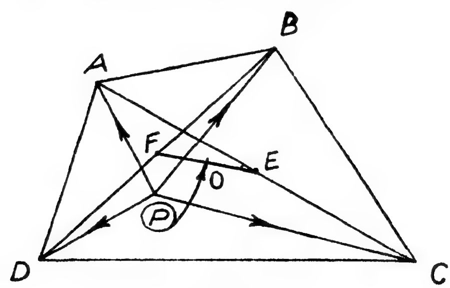


FIG. 119

192. *ABCD* is a trapezium of which the side *AB* is parallel to *DC*. Three forces acting on a particle can be represented in magnitude and direction by *AC, BD* and *CB*. Show that they are in equilibrium.

193. If three forces act at a point to keep it at rest and if two of them can be represented along the two diagonals of a square and also in magnitude by their lengths, show that twice the length of a side of the square represent the magnitude of the third force.

194. Three forces P, Q, R act at a point and are in equilibrium. If P be 100 lbs. and be directed towards north, and if Q and R act towards 70° south of east and 80° south of west respectively, find the magnitudes of Q and R.
Ans. $Q = 3.472$ lbs.; $R = 6.84$ lbs.

195. A mass of 30 lbs. is suspended from the ceiling of a laboratory room. A spring balance is attached to a horizontal cord to pull the mass. If the suspending cord makes an angle of 30° with the vertical, what will be the amount of pull recorded by the balance? What is the tension in the suspending cord? *Ans.* 69.28 lbs.; 34.64 lbs.

196. In the foregoing example if the mass be drawn 60°, establish the relation between the magnitude of the weight and the tension.
Ans. 1 : 2.

197. Forces 3, 4, 5 keep a body in equilibrium, draw a diagram to show their lines of action.

Any of the three forces is not equal to the sum of the other two. Therefore, the forces cannot be parallel and hence they must meet at a point.

Draw a triangle ABC of which $AB = 3$, $BC = 4$ and $CA = 5$ (in some definite scale). Take any point O, from O draw a straight line $OP \| AB$, $OQ \| BC$ and $OR \| CA$. Then OP, OQ and OR are the lines of action of the three forces.

198. One end of a rope l feet long is fixed to a vertical telegraph post standing on the ground, and a man pulls at the other end with a given force. Show that if the rope is fixed at a height of $\frac{1}{\sqrt{2}}$ foot from the ground, the man will have the best chance of overturning the post.

199. A crane is lifting a bridge-frame with the help of a chain with its ends fixed with the two ends of the frame keeping it in a horizontal position. Prove that the chain cross-section can be reduced by increasing the length of the chain.

200. Find the forces acting through the members AB and BC of the triangular roof-frame ABC. Then span AC is 20 ft. The members AB and BC make angles of 30 and 45 degrees with AC. A load of 1 ton acts vertically at the appex B. *Ans.* AB — 1640 lbs.
BC — 2008 lbs.

201. BC is a cantilever beam (Fig. 120). A weight of 5 tons is suspended from the free end. A tie-rod AC is fitted to share the load with BC. Find the forces induced in the members BC and AC.

Solve the problem with different methods—Vector diagram, Method of resolution, Principle of moments—and check the results with the help of Lami's Theorem.

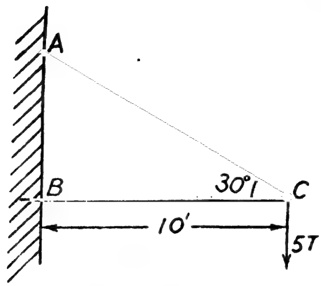


FIG. 120

Ans. BC — 8.66 tons,
AC — 10 tons.

202. Fig. 121 represents a structure under the load of 2000 lbs. as shown. Assuming the force at F as horizontal determine the H and V components of the reaction at E. What are the components at A, B and C?

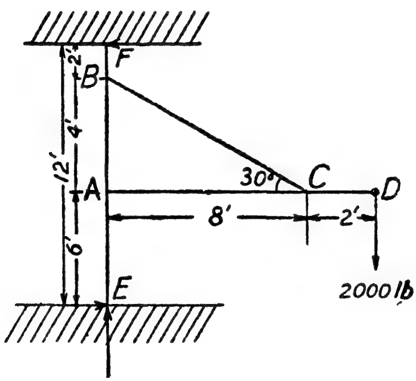


FIG. 121

203. Illus Exam. 98 as directed.

204. In a crank and connecting rod mechanism of a steam engine (Fig. 122), the pressure on the piston is 5000 lbs. At an instant the angle between the crank and the piston travel line is α. Find the loads in the connecting rod when α is equal to 30°, 60°, 90° and 100° respectively.

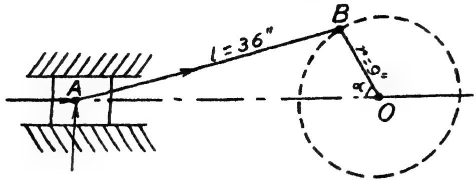


FIG. 122

Ans 5041, 5123, 5165 and 5129 all in pounds.

205. A Bell-crank lever—angle between the arms is 100°. Bigger arm $AB = 8$ ins. The ratio between the effective arm lengths is 2 . 3. A linkage is fitted to the end C and at position it is at right angles to BC. If a force of 1000 pounds be applied at the end A, normal to AB, determine the force transmitted to the link. What is the reaction at B? (Fig 123)

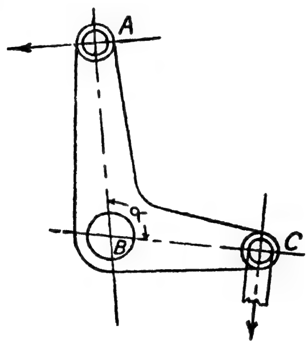


FIG. 123

Ans. 1500 lbs., 1941.8 lbs.

206. Find the resultant of a system of forces in fig. I in the block (P. 241).

Ans. 50 lbs., 2.8 feet from the 20-lbs. force.

207. Find the resultant of a system of force in fig. II in the block (P. 241).

Ans. 105.5 lbs., the line of action cuts the line AB at a distance of 2.4 ft. from B making an angle of 32.6° with AB.

208. Find the resultant of a system of forces in fig. III in the block (P. 241).

Ans. 8.34 lbs., 78.2° inclined to BA, cuts BA at a distance of 1.4 ft. from B.

209. Find the resultant of a system of forces in fig. IV in the block (P. 241).

210. Find the resultant of a system of forces in figure V in the block (p. 241).

The side of a small square in the block represents 1 foot and the digits, against the lines with arrows denoting the lines of action of the forces, give the magnitudes of the forces in pounds.

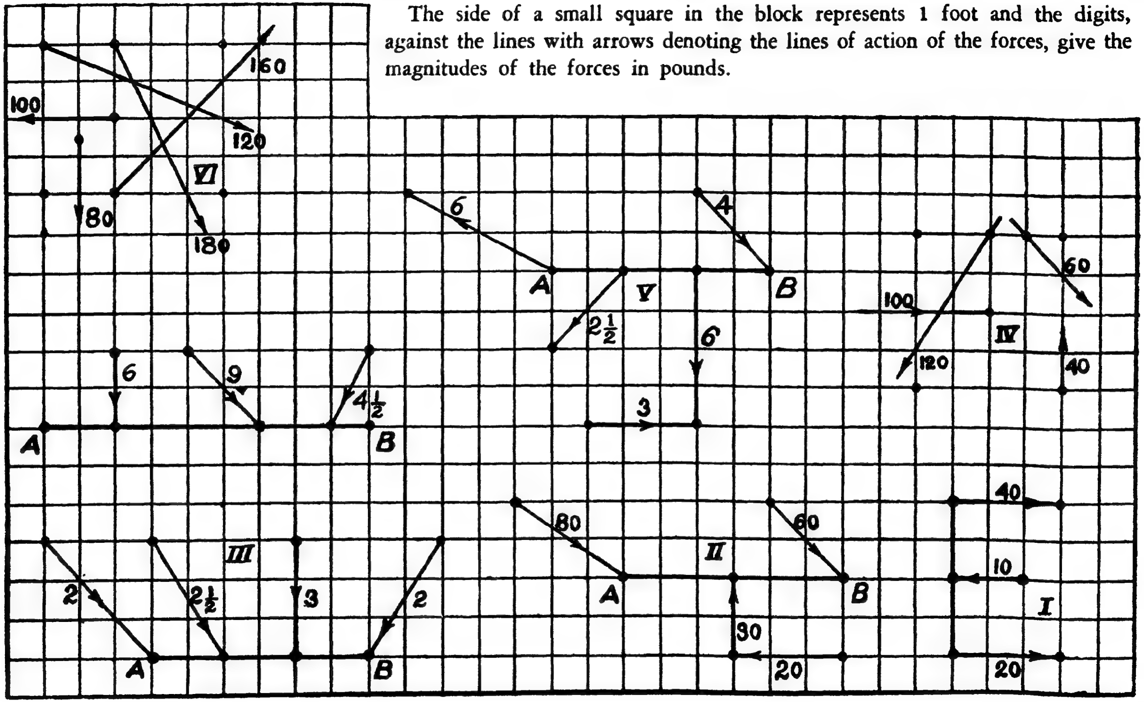


FIG. 124 to 130

211. Find the resultant of a system of forces in figure VI in the block (p 241)

212. Find the resultant of the remaining system of forces in the block (p 241)
Ans 16 96 lbs at a distance of 4 107 ft making an angle of 75 1° clockwise from A

213 Determine the resultant of the system of forces in pounds given in Fig 131

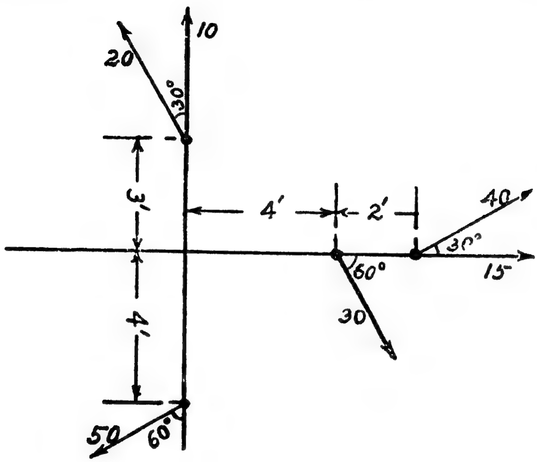


Fig 131

Ans 19 1 lbs, at a distance 9 06 ft from the origin to produce a clockwise moment, so the V component becomes negative

214 *ABCD* is a four bar linkage (Fig 132) At the position the angle $\alpha = 30°$ and *CD* is at right angles to *BC* The line of action of the force at *B* is at 90° with *AB* and $\alpha_1 = 45°$ Find the force Q to keep the system in equilibrium

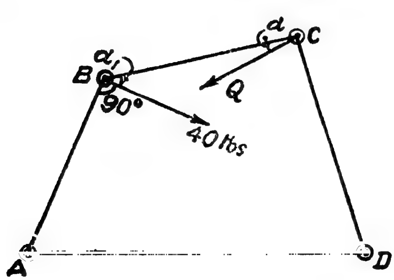


Fig 132 *Ans* 65 2 lbs.

215. A horizontal rod simply rests on two supports at the ends (Fig. 133). An inclined thrust of 100 lbs. is applied at a distance of 3.5 feet from the end A as shown. Find the vertical reactions at the supports. If the rod is hinged at the end A and freely rests on a smooth bearing at the other end, find the reactions at the bearing and at the hinge in magnitudes and directions.

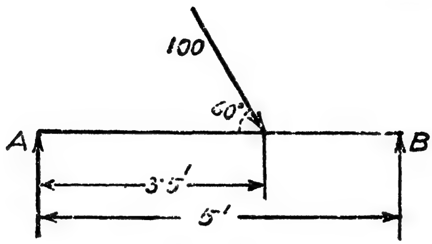


FIG. 133

		A	B
Ans.	(1)	25.96 lbs.	60.62 lbs.
	(2)	56.35 lbs.	60.62 lbs.
		27.45° with	vertical.
		vertical clockwise.	

216. A wall-crane ABC (Fig. 134) is fitted to raise a maximum load of 5000 lbs. by means of a rope passing over pulleys at B and D. Find the forces in AB and BC.

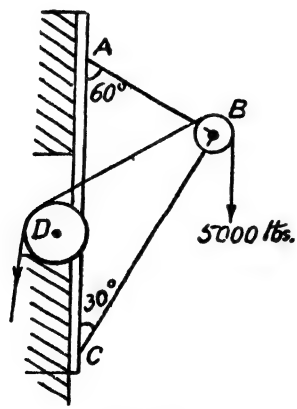


FIG. 134

Ans. AB — 2500 lbs. BC — 4330 lbs.

217. A slewing jib-crane (Fig. 135) can turn about the axis AB. If the weight of the crane be 5000 lbs. and act at a distance of 5 feet from AB to the right, and if the load lifted be of amount 5000 lbs., find the reactions of the bearings at C and D.

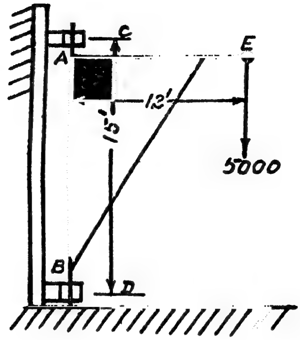


FIG. 135

Ans 5666 6 lbs. horizontal
At C pull
At D push

218. A crane shown in Fig 136 has the maximum lifting capacity of 5000 lbs. Find the reactions of the supports at A and B, and also the resistance offered by the members when full load is applied A is at a level 12 feet above that of B.

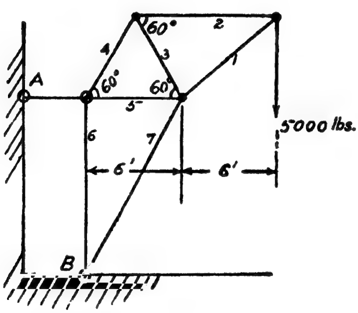


FIG. 136

Ans. $R_A = 5000$ lbs, $R_B = 7071$ lbs., $\theta = 45°$.

1	2	3	4	5	6	7
6070	4585	2297	2297	3264	5000	11200
C	T	C	T	C	T	C

219. Loads in pound weight are applied on different points of a crane shown in Fig. 137. Find the reactions of the supports and the loads shared by the members.

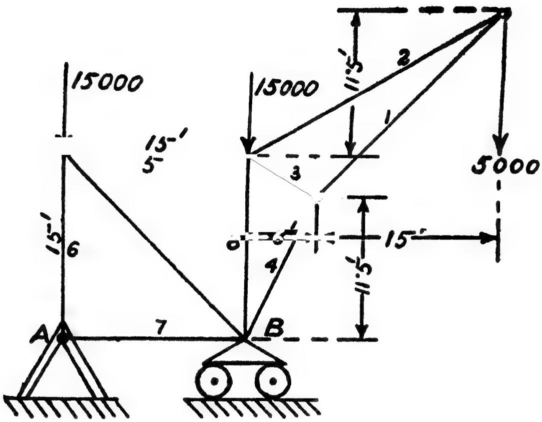


FIG 137

Ans $R_A = 8000$ lbs, $R_B = 27000$ lbs

1	2	3	4	5	6	7	8
15710	7568						
C	T						

220. The weight of a 21 feet ladder (Fig 138) is 50 lbs and it leans against a smooth vertical wall making an angle of 45° with the horizontal The lower end of it is fixed against a step as shown A man weighing 10 stones is climbing up the ladder Find the reactions of the wall and the step when the man is at a distance of $\frac{2}{7}$ of the length of the ladder Take the weight of the ladder to act through a point at the middle of its length

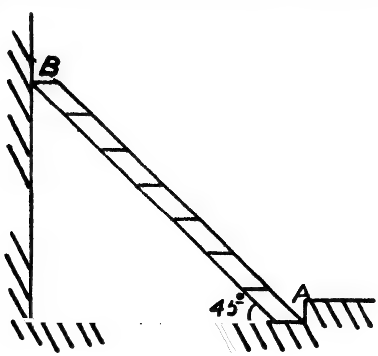


FIG. 138

Ans 65 lbs normal to the wall
200 8 lbs — 71° with floor.

221. An iron ball weighing 10 lbs. is placed on two inclined planes fixed as shown (Fig. 139). Find the reactions at *A* and *B*, the points of contact, assuming the line of action of the weight passes through the centre of the spherical body. If the diameter of the sphere is 4 inches and if the planes are kept in the position by drawing the end of the side *A* in a direction at right angles to it and if this side as 4.5 inches long, find the force of drawing.

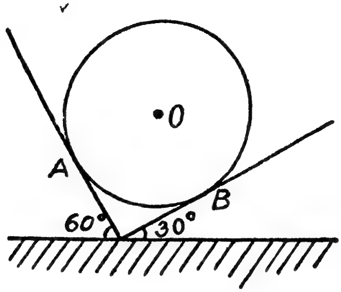


FIG. 139

Ans. 5 lbs., 8.66 lbs., 1.63 lbs

222. The system of Five vertical parallel forces are acting as shown (Fig. 140). All the forces given are in pounds. If the force *F* is 650 lbs. and acts for the first instance in the downward direction and for the next time in the upward direction, find for both the cases the magnitudes, directions and positions of the resultants.

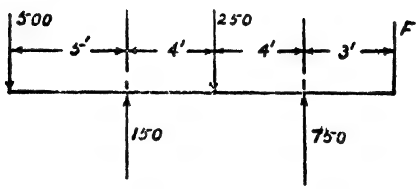


FIG. 140

Ans. 500 lbs downward at a distance of 3.3 ft. to the right of the extreme left force.
800 lbs. upward at a distance of 23.3 ft. to the right of the extreme left force.

223. Find the magnitude, direction and position in space of the force *F* in Fig. 140, if the five forces form a balanced system.

Ans. 150 lbs. downward 55 ft. to the right of the extreme left force.

224. Determine the magnitude, direction and the position in space of the resultant of the five parallel forces (Fig. 140) if $F = 330$ lbs. downwards and acts at a distance of 25 feet from the extreme left force. The position is to be determined with respect to the position of the 50-lb. Force.

Ans. 180 lbs. downwards in the line with the reference force.

225. The six parallel forces, acting at A, B, C, D, E and F along a straight line at relative distances as shown in Fig. 141, form a balanced system. Find the magnitudes and directions of the two unknown forces at C and F.

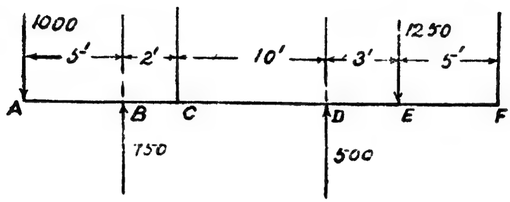


FIG. 141

Ans. 680 lbs. upward, 320 lbs. upward.

226. Find the resultant of the two parallel forces and two couples as shown in Fig. 142. The digits by the side of the lines of action of the forces denote the magnitudes of the forces in pound weight.

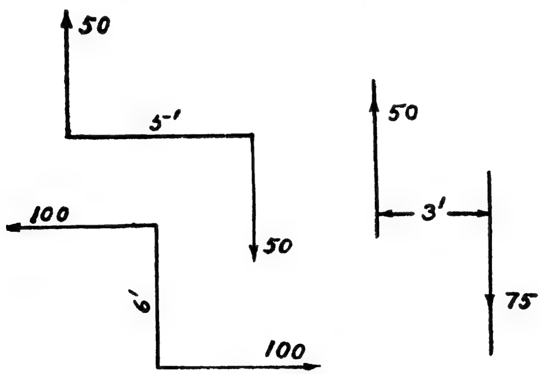


FIG. 142

Ans. 25 lbs. in a direction opposite to the 50-lb. force at a distance of 5 ft. to the left of it.

227. Two steel balls A and B weighing 8 and 4 ounces respectively are connected by a fine 6-in. silk thread and placed on the smooth surface of a

roller 6 inches in diameter so that the plane passing through the thread and the centres of the balls is at right angles to the axis of the roller Find θ_1 and θ_2 when the system is in equilibrium Find also the reactions (Fig 143)

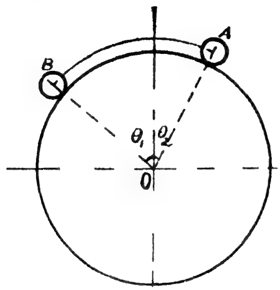


FIG 143

Ans $\theta_1 = 84\ 75°$ $\theta_2 = 29\ 85°$
$R_A =$ 6 46 lbs $R_B =$ 36 lbs

228 Prove that if three forces can be represented in magnitudes, directions and positions by the sides of a triangle taken in order, the forces will form a couple

229 A bridge girder (Fig 144) carries three equal loads of 8 tons as shown The inclined members are at 45° with the horizontal Find the forces in the members

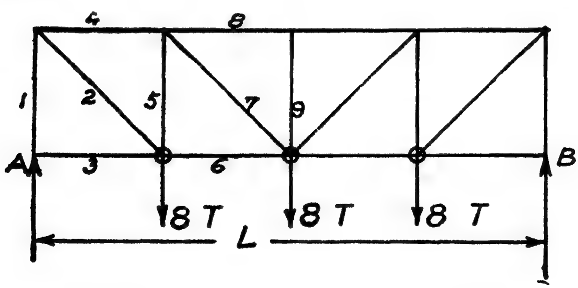


FIG 144

Ans Forces are given in pounds

1	2	3	4	5	6	7	8	9
12 C	0	17 T	12 C	4 C	12 T	5 66 T	16 C	0

230. In a bridge truss (Fig 145) the lower horizontal member is equally divided into three parts The inclined members have inclination of 45° with the horizontal member Two equal loads of 8 tons are applied at the two

inner joints as shown. Find the forces in the members by the method of section or otherwise.

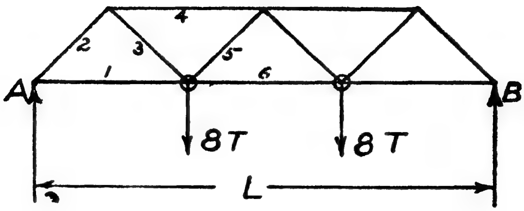


FIG. 145

Ans. Forces are given in pounds.

1	2	3	4	5	6
17920 T	25260 T	25260 C	35840 C	0	35840 T

231. A roof truss with loads in pounds is shown in Fig. 146. Determine the reactions at A and B and also the forces acting through the members, 1, 2, 3 and 4.

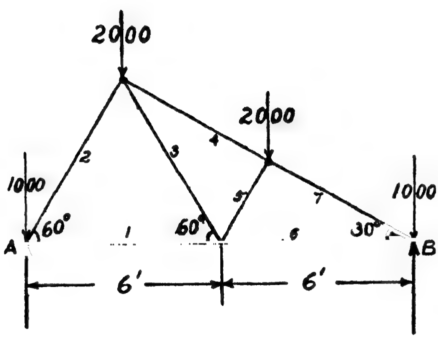


FIG. 146

Ans. $R_A = 3250$ lbs. $R_B = 2750$ lbs.

1	2	3	4
1125 lbs.	2598 lbs.	866.2 lbs.	2000 lbs.
T	C	T	C

232. A roof truss is shown in Fig. 147; the loads are in pounds weight. Find the vertical reactions at A and B. Also find by method of section the forces acting through the members 2, 3 and 4.

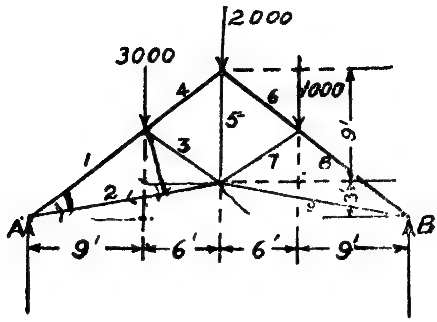


FIG. 147

Ans. $R_A = 3400$ lbs., $R_B = 2600$ lbs.

2	3	4
5800 lbs.	2450 lbs.	4750 lbs.
T	C	C

233. If the truss in the previous problem is fixed at the ends with the supports find the reaction at A, also the forces acting through the members of the truss.

Ans. Horizontal component 10 lbs.
The change is negligible.

234. The distribution of loads are given in the diagram of the roof truss of Fig. 148 ; find the reactions at the supports assuming them to be vertical and also the magnitudes of the forces acting in the members. Find also whether they are in tension or compression.

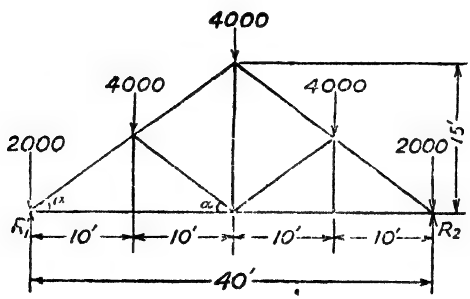


FIG. 148

235. In the cantilever girder (Fig. 149) loads are acting as shown. Determine the reactions at the supports and also the magnitudes and natures of forces acting in the members.

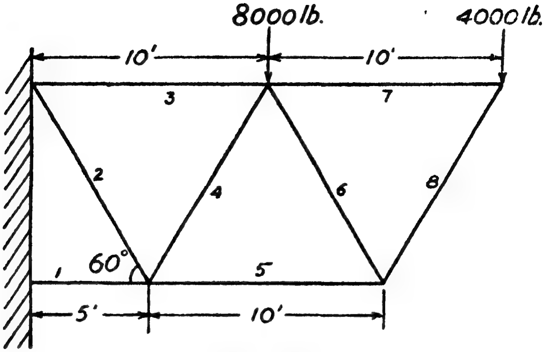


FIG. 149

Ans.	8	7	6	5	4	3
	4620(C)	2310(T)	4620(T)	4620(C)	13520(C)	11400(T)

236. A frame shown in Fig. 150 has loads in pounds weight at different joints as given in the diagram. It is supported on fixed pins at the base. Find the vertical and horizontal components of the reactions at the support. Find also the forces acting through different members.

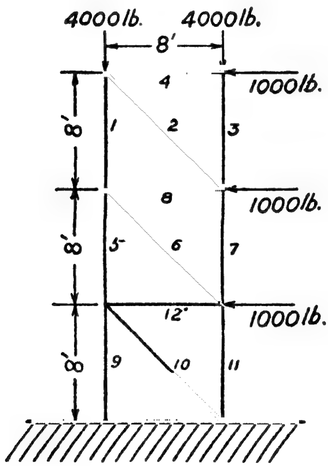


FIG. 150

Ans.	1	2	3	4	5	6	7	8	9
	5000	1415	4000	1000	7000	2828	3000	2000	10000
	C	T	C	C	C	T	C	C	C

10	11	12
4242	1000	3000
T	C	C

237. Eight forces are in equilibrium (Fig. 151). The two known forces of 1000 lbs. each are shown in magnitudes and directions. The lines of action of the six unknown forces—Nos. 1, 2, 3, 4, 5 and 6—are given They act at the corners of a rectangular prism. Find the unknown forces.

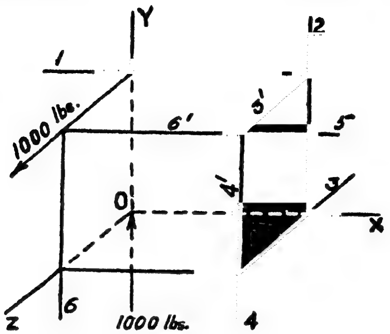


FIG. 151

238. Fig. 152 is a cube of sides equal to 10 inches If equal forces of 5 lbs. each act along the edges as shown by arrow-heads, find the resultant of the system.

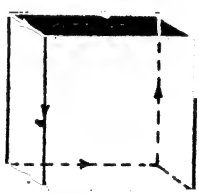


FIG. 152

239. Section of a tie-rod is shown in Fig 153 The stress intensity varies from O at DF to maximum 2 tons per square inch at AB Find the centre of the stress with respect to the sides AB and AM

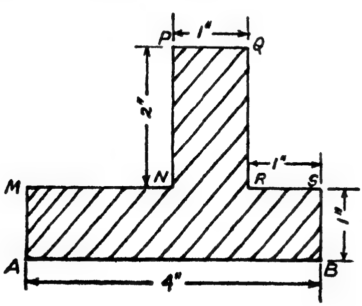


FIG. 153 *Ans.* $\frac{2}{3}$ inch, $2\frac{1}{12}$ inches.

240. A Theodolite weighing 15 lbs. is fitted on a tripod stand on a horizontal floor. The legs are of equal lengths and form equal angles with each other. The angle made by each leg with the floor is 60°. Determine the forces acting in each leg. What is the length of the side of the triangle formed by joining the three points of rest of the legs?

Ans. 5.77 lbs.

241. Three rods are hinged at C from which a load $W = 500$ lbs. is suspended (Fig. 154). The rods AC and BC are horizontal and make 45° with the fixing wall. The member BC which is fixed midway below between A and B with the wall also makes an angle of 45° with the wall as shown. Find the forces acting through the three members. Find also whether they are in tension or compression.

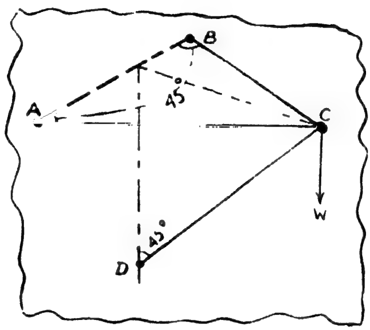


FIG. 154

Ans.	DC	AC BC
	707 lbs.	353.6 lbs.
	C	T

242. Three balls of equal weights and radii are placed on a rough floor so that they are in contact with each other. A fourth ball of the same radius and weight W is placed on the top of the previous three balls. Compute the normal reaction of each of the three lower balls on the upper one.

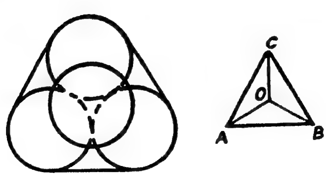


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FIG. 155

Ans. .385 W

243. The lower three balls are tied together by means of a cord around their equatorial plane. Find the tension in the cord caused by the weight of the upper ball assuming that there is no initial tension in the cord (Fig. 155).

Hints for Problems 242 and 243 :—

If the centres of the lower three balls are joined, the three straight lines will form an equilateral triangle. If, again, a vertical line be drawn through the intersecting point of the medians (Fig. 155-I) of this triangle, it will pass through the centre of the upper ball, because any point on the vertical line is equidistant from the three centres of the three balls on the floor, and the distances of the centre of the upper ball from those three centres are equal. By joining all the centres we get an equilateral triangular pyramid. Therefore, if r be the radius of the circle, each side of the pyramid is equal to $2r$, and the length of the median of the base triangle (and of all the triangles formed) $= 2\ r \cos 30 = r\sqrt{3}$.

Hence, the angle made by the vertical through the centre of the upper ball and the line joining the centre of the upper ball with the centre of any three lower balls will be such that, $\frac{2}{3}\ r\ \sqrt{3} \div 2\ r = \sin\ \theta$, where θ is the angle.

$$\text{Therefore, } \sin\ \theta = \frac{1}{\sqrt{3}},\quad \cos\ \theta = \sqrt{\frac{2}{3}}\ \text{ and } \tan\ \theta = \frac{1}{\sqrt{2}}$$

$$\text{Hence, reaction} = \frac{W}{3} \div \cos\ \theta = \frac{W}{\sqrt{6}}$$

$$\text{Horizontal component} = \frac{W}{3}\ \tan\ \theta = \frac{W}{3\sqrt{2}}$$

$$\frac{T}{\cos 60} = \frac{W}{3\sqrt{2}.\ \sin 60}\quad \text{or, } T = \frac{W}{3\sqrt{6}}$$

N.B. In the results of the problems on trusses, the letters C and T indicate compressive and tensile loads respectively.

CHAPTER VIII

FRICTION AND LUBRICATION

198. In cases of motion it is generally found that one surface is sliding or rolling over another surface. These surfaces may be generally of two different kinds,

(I) Smooth and (II) Rough.

Perfectly smooth surface is an impractical thing. In a theoretical perfectly smooth surface the reaction due to any force applied on it is always directed normally ; any component along the surface is impossible due to its smoothness.

199. In cases of rough surfaces when one surface tends to slide over the other, a force of resistive kind comes into action. Its action is to resist any sliding motion between the two surfaces in contact. This property of two surfaces in contact by virtue of which a resistance is offered to any sliding motion between them is called *Friction.*

Force of friction acts as equilibrant of forces acting on many bodies at rest on the surface of the earth. The resultants of the applied forces and the frictional resistances acting on a body at rest and the reaction of the plane on it are assumed to be co-planer and concurrent, as if, the force system is acting on a particle.

200. The force due to friction as described above gradually increases with the application of the sliding force. But there is a limit to this action. Up to that limit the force due to friction adjusts itself to maintain equilibrium of the system. This limiting value of the frictional force is equal to the least force that is required to be applied parallel to the surfaces in contact to cause one surface slide over the other and it is called *Limiting Friction.*

201. If the force of friction is represented by a letter F and if the mutual normal pressure between the two surfaces in contact be represented by R, then the ratio, $\frac{F}{R}$ is called the *coefficient of friction,* and is denoted by μ. Therefore, $F = \mu R$ ············*Eq.* 103

202. It is found that the friction at the starting, *i.e.*, the statical friction is greater than the friction in motion.

203. The friction is approximately governed by the following three laws :

I. *The limiting friction is directly proportional to the mutual normal pressure between the two surfaces in contact.*

II. *It is independent of the area of surfaces in contact.*

III. *It is independent of the velocity of sliding.*

204. $F = \mu R$. Though F does not depend on the area of surfaces in contact, μ, *i.e.*, $\frac{F}{R}$ depends on the nature of the surfaces which includes the gradation of smoothness, lubrication, composing materials of the bodies, etc.

The above laws are approximately true when the intensity of normal pressure is moderate and when the sliding speed is low.

205. *Friction angle, angle of friction, the limiting angle of resistance or the limiting angle of friction.*

If a body rests on a horizontal plane and a horizontal force is applied to move the body, the forces acted by the plane on the body are the reaction due to the weight of the body, which is at right angles to the plane and the force that appears due to friction in opposition to the motion. With the application of the force, the force due to friction increases up to the value of limiting friction, when the body just begins to move. This limiting friction, F, and the reaction, R, have a resultant S, that makes an angle λ with the direction of R as is shown in the diagram (Fig. 156). This extreme inclination of the resultant of the normal reaction and the force due to friction with the direction of the former is called the *friction angle, angle of friction, the limiting angle of resistance or the limiting angle of reaction.* In the diagram for vector addition (Fig. 156), $\frac{F}{R} = \tan \lambda$, which is also equal to μ. That

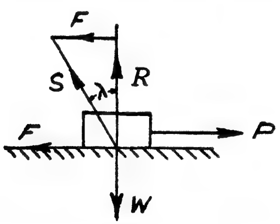


FIG. 156

is, *the coefficient of friction is equal to the tangent of the friction angle.*

206. Angle of Repose, *i.e.*, the extreme inclination of a plane on which a body may rest unaided.

Due to the inclination of the plane the natural tendency of a body resting on it is to slide down along the plane. But owing to the appearance of the frictional force the body may remain at rest without sliding up to a limit of the inclination.

The only force applied by the body on the plane is its own weight W (Fig. 157). Let α be the required inclination. To keep the body at rest the force applied by the plane on the body must be equal and opposite to W. That force S, must, therefore, be the resultant of the limiting friction F, which is equal to $W \sin \alpha$ and the normal reaction R, which is equal to $W \cos \alpha$. According to the previous article, S may have the maximum inclination with the direction of R by λ. From the diagram it is found that the angle between R and S, is equal to α. Therefore, the maximum inclination may attain a value equal to λ to satisfy the condition. Hence, the friction angle may also be defined as the greatest inclination of a plane on which a body may rest unaided. This maximum inclination of a plane is called the *angle of repose.* Thus, the angle of friction is equal to the angle of repose.

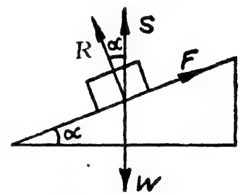


FIG. 157

207. *Deduction of formulae to establish relations between the force required for sliding a body up or down an inclined plane and its weight in different cases.*

I. *Force (general).*

Let P be the force (Fig. 158) required to be applied in a direction making an angle θ with the horizontal direction to start the motion of the body up a plane whose inclination is α. Just before the body begins to move the three forces acting on the body (the moving force P, its weight W and the resultant of the forces, S, exerted by the plane on the body) are in equilibrium. If a vector diagram is drawn for these three forces in equilibrium, it will be a triangle *abc* as shown

in the diagram (Fig. 158-*b*). The direction *bd* represents the direction of the normal reaction. In the triangle *abc*,

$$\frac{P}{W} = \frac{\sin(\alpha + \lambda)}{\sin acb} = \frac{\sin(\alpha + \lambda)}{\sin(180 - abc - bac)}$$

$$= \frac{\sin(\alpha + \lambda)}{\sin\{180 - (\alpha + \lambda) - (90 - \theta)\}} = \frac{\sin(\alpha + \lambda)}{\sin\{90 + \theta - (\alpha + \lambda)\}}$$

$$= \frac{\sin(\alpha + \lambda)}{\cos\{\theta - (\alpha + \lambda)\}} \qquad \cdots\cdots \quad Eq.\ 104$$

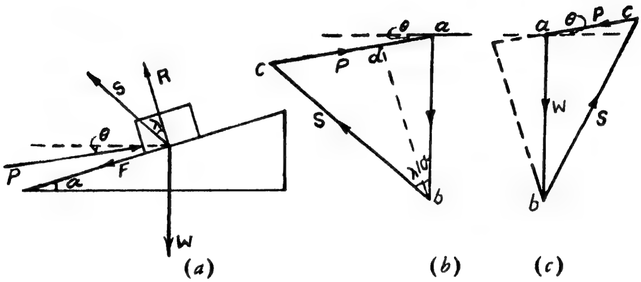


FIG. 158

If the force be applied so as to move the body down the plane, the ratio, (Fig. 158-*c*),

$$\frac{P}{W} = \frac{\sin(\lambda - \alpha)}{\cos\{\theta + (\lambda + \alpha)\}} \qquad \cdots \quad Eq.\ 105$$

II. *Force parallel to the plane.*

When $\theta = \alpha$, *i.e.*, when the force is in a parallel direction with the plane, by drawing the vector diagram and also from calculation it is found that

$$\frac{P}{W} = \frac{\sin(\alpha + \lambda)}{\cos\lambda}\text{, when moving up} \qquad \cdots\cdots\cdots\cdots Eq.\ 106$$

and $$\frac{P}{W} = \frac{\sin(\lambda - \alpha)}{\cos\lambda}\text{, when moving down} \qquad \cdots\cdots\cdots\cdots Eq.\ 107$$

III. *Horizontal Force.*

When $\theta = 0$, *i.e.*, when the direction of the force is horizontal,

by drawing the vector diagram and also from calculation it is found that

$$\frac{P}{W} = \tan(\alpha + \lambda), \text{ when moving up} \quad \cdots\cdots\cdots\cdots Eq.\ 108$$

and $$\frac{P}{W} = \tan(\lambda - \alpha), \text{ when moving down} \quad \cdots\cdots\cdots\cdots Eq.\ 109$$

IV. *Least Force.*

When P has got the least value, by drawing the vector diagram it is found that

$$\frac{P}{W} = \sin(\alpha + \lambda), \text{ when moving up} \quad \cdots\cdots\cdots\cdots Eq.\ 110$$

and $$\frac{P}{W} = \sin(\lambda - \alpha), \text{ when moving down} \quad \cdots\cdots\cdots\cdots Eq.\ 111$$

Also from calculation, $\frac{P}{W}$ will be least, when $\frac{\sin(\alpha + \lambda)}{\cos\{\theta - (\alpha + \lambda)\}}$ will be least, *i.e.*, $\cos\{\theta - (\alpha + \lambda)\}$ will be maximum, *i.e.*, when $\theta - (\alpha + \lambda) = 0$, or, $\theta = (\alpha + \lambda)$, and then, $\frac{P}{W} = \sin(\alpha + \lambda)$. The direction of the force P will, of course, make an angle λ with the plane.

Similarly, the least force to move a body down the plane will be such that $\frac{P}{W} = \sin(\lambda - \alpha)$.

In this case there may be two solutions, (a) when λ is greater than α and (b) when it is less than α. The first case is a normal one and in case (b) where λ is less than α, the direction will be upwards because in that case the downward component of W, which is $W \sin\alpha$, is greater than F and the quantity of force P is required just to maintain equilibrium.

In both the cases of (a) and (b), the direction of P will make an angle λ with the plane.

208. The application of the relations established between the effort and the load in motion on an inclined plane is ordinarily found in all the machine parts or machines having screw-nut mechanism. If, for

example, the case of a Jack-Screw, which is generally made of square thread, be taken, the portion which stands for the nut in the mechanism is ordinarily made fixed. The threaded rod being rotated will silde up or down the plane of the thread of the nut which can be compared with an inclined plane. The description of machines with screw-nut mechanism will be found in the chapter of machine (Chapter XII).

209. Screw and Nut in transmiting Motion and Power.

I. *Square-threaded Screw.*

Generally a square-threaded screw and nut is used in transmitting motion and power. The diagram (Fig. 159) represents a line sketch for a square-threaded screw. The mean diameter is the *effective diameter* of the screw. For one rotation the nut will move by a pitch-length in case of a single-threaded screw in a direction parallel to the axis. From any point on a thread surface to the corresponding point on the next thread surface measured along a direction parallel to the axis is termed as the *pitch* of the screw. One thread plus one

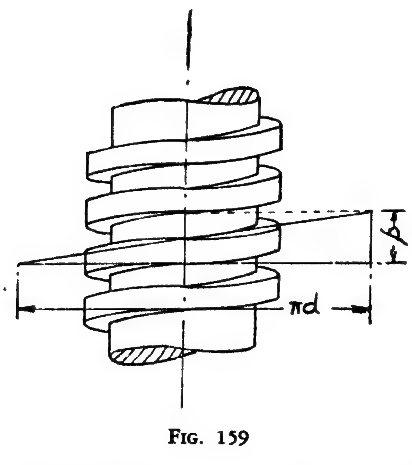


FIG. 159

gap is equal to one pitch-length. It is to be marked that the line of action of the force applied on a nut or screw always lies on a plane at right angles to the axis of rotation. It is also to be marked that the thread surface on which the sliding takes place can be taken as an inclined plane whose $\tan \alpha = \dfrac{p}{\pi d}$, where α is the angle of inclination,

p the pitch and d the mean diameter. Again, from *Deduction III*, Art. 207, when the part moves up $\dfrac{P}{W} = \tan(\alpha + \lambda)$

$= \dfrac{\tan\alpha + \tan\lambda}{1 - \tan\alpha . \tan\lambda}$. Substituting the values of $\tan\alpha$ and $\tan\lambda$,

$$\frac{P}{W} = \frac{p + \mu \pi d}{\pi d - \mu p} \qquad \cdots\cdots \quad Eq.\ 112$$

Here P is the horizontal force or effort that is required to be applied at the mean radius of the rod to create motion in the moving part and W is the load, *i.e.*, the vertical downward force on the moving part. By the principle of work, the work done by the effort should be equal to the work obtained at the load, *i.e.*, $P \times \pi d = W \times p$. But in all machinery it is found that $P \times \pi d$ is not equal to $W \times p$. There is a loss due to the resistance offered by the machine parts. Therefore, the loss $= P \times \pi d - W \times p$.

Again, in case of square-threaded screw, if A be the total bearing area of the screw thread, and w_n be the normal pressure intensity on the surface, then $w_n = \dfrac{W}{A} \cos\alpha$.

Frictional resistance per unit area $= \mu\, w_n$. If T be the horizontal torque applied to rotate the screw by one turn, then, by the principle of work,

$$2\pi T = \mu \left(w_n . A . \frac{2\pi r}{\cos\alpha} + W.p + \mu W \sin\alpha \right)$$

$$= 2\pi \mu\, w_n . A . \frac{r}{\cos\alpha} + 2\pi W r \tan\alpha$$

Wrong.

$$\therefore \quad T = \mu \frac{W}{A} \cos\alpha . A . \frac{r}{\cos\alpha} + W r \tan\alpha$$

$$= \mu W r + W r \tan\alpha$$

$$= W r (\mu + \tan\alpha) = W r (\tan\lambda + \tan\alpha) \quad \cdots\cdots Eq.\ 113$$

II. *V-threaded screw.*

In case of V-thread screw let the angle between the two surfaces of a screw thread be θ as shown (Fig. 160). Here $w_n = \dfrac{W \cos\alpha}{A \cos\frac{\theta}{2}}$

(resolving for the obliquity of the surface at right angles to the plane and the axis respectively). Therefore,

$$T = \mu \frac{W \cos \alpha}{A \cos \frac{\theta}{2}} . A . \frac{r}{\cos \alpha} + W r \tan \alpha$$

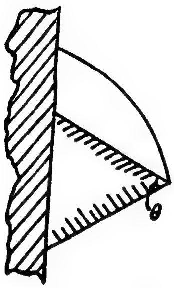


FIG. 160

$$T = W r \left(\mu \frac{1}{\cos \frac{\theta}{2}} + \tan \alpha\right)$$

$$= W r \left(\frac{\tan \lambda}{\cos \frac{\theta}{2}} + \tan \alpha\right)$$

Eq. 114

In case of Whitworth thread $\theta = 55°$, and

$$T = W r (1.13 \tan \lambda + \tan \alpha) \qquad Eq.\ 115$$

In case of Sellers thread $\theta = 60°$, and

$$T = W r (1.15 \tan \lambda + \tan \alpha) \qquad Eq.\ 116$$

210. *Force required to create rotation or to bring to rest a rotating body such as the wheels of vehicles.*

Generally after a heavy shower of rain it is often found that though force is applied to start a train at rest, the train does not move. Its driving wheels rotate about the axes sliding or skidding over the surfaces of the track but do not roll. The same phenomenon will be observed when a tram-car tries to run up a slope after a heavy shower. The reason is that the coefficient of friction between the wheels and the track is reduced to a very low value and is not sufficient enough due to the slippery condition of the track, to allow the necessary amount of the effective force to be developed that should be applied to drag the mass. If the wheel slides over the track surface no point on its circumference can be taken as stationary ; whereas, if the wheel rolls, then, at any instant the point of the wheel in contact with the track may be considered as the (instantaneous) centre about which the whole wheel rotates. So the coefficient of friction should be such that the point of the wheel in contact with the track may be made to adhere

for the instant against the track surface. In these cases the coefficient of friction is called the *coefficient of adhesion.*

In cases of stopping a running vehicle brake-blocks are generally provided so that the blocks may be pressed against the inner or outer surface of the wheels to create a resistance against the rotary motion of the wheels. It is often found that after a definite normal pressure the wheels cease to rotate but begin to skid, specially when the track is slippery. One can sometimes notice it when one runs on a bicycle over a slippery road just after the watering of the streets. This shows that the normal pressure on the wheel may be increased to any amount whatsoever, but beyond a certain value this does not help to stop the vehicle in any way. The force that can be applied effectively has got a limit and it depends on the value of the coefficient of friction. Thus, in case of creating a motion as well as in case of stopping a motion, the force that can be applied effectively depends on the value of μ. It is also a function of the weight on the driving wheels, because the force that can be applied, $F = \mu R$. Effective brake-force is the measure of the maximum force that can be applied to create motion.

211. Work and Friction. If the working part of a machine rotates due to the exertion of a force P lbs. at a distance of R inches from the axis, then, the total work done per minute is equal to $2\pi PRN$ in. lbs. where N is the number of rotations per minute. But if there is a constant resistance of F lbs. (which generally arises from the friction in the bearings) at a distance of R_1 inches from the axis against the motion, then, of the total work done, $2\pi PRN$ in. lbs., the amount of work, $2\pi FR_1N$ in. lbs. will be spent in overcoming the frictional resistance. The effective work done per minute will be $2\pi N(RP - R_1F)$ in. lbs. Therefore, the effective horse-power of the machine,

$$H.P. = \frac{2\pi N(RP - R_1 F)}{12 \times 33000} \qquad \cdots\cdots\cdots\cdots Eq.\ 117$$

Illus. Ex. 104. ***Minimum horizontal force to drag a block of wood weighing 100 lbs. over an iron track is 20 lbs. What is the coefficient of friction between the wood and iron? If the direction of the force be changed to 30° to the horizontal, find the force.***

(*i*) $\mu \times 100 = 20$ **lbs. Therefore,** $\mu = \dfrac{20}{100} = .2$

(*ii*) If P be the force, P can be resolved into P sin 30 and P cos 30. Due to the action of P sin 30 the normal pressure is reduced to $100 - P$ sin 30. Therefore, the horizontal component,

$$P \cos 30 = \mu (100 - P \sin 30) = .2 \left(100 - \frac{P}{2}\right) = 20 - \frac{P}{10}$$

$$\text{or, } P\left(\frac{1}{10} + .866\right) = 20. \qquad \therefore \quad P = 20.7 \text{ lbs.}$$

Illus. Ex. 105. *A train with its locomotive weighs 135 tons. It starts from a station and the velocity is raised to 45 miles per hour in 55 seconds. If the coefficient of adhesion be $\frac{1}{7}$, find the necessary weight on the driving wheels. Compare the times required to stop the train from the speed with brakes on the driving wheels only and with brakes on every wheel in the train.*

$$\text{The acceleration} = \frac{66}{55} = 1.2 \text{ feet per sec. per sec.}$$

$$\therefore \quad \text{the accelerating force} = 1.2 \times \frac{135}{32.2} = 5.31 \text{ tons.}$$

(*i*) For the application of this force without causing any slip in the driving wheels the weight of the mass on the driving wheels will be $7 \times 5.31 = 37.17$ tons (μ being equal to $\frac{1}{7}$).

(*ii*) If the brakes be applied on the driving wheels only the maximum force that can be applied to stop the train is 5.31 tons. The accelerating force is equal to the retarding force. Therefore, the time required is 55 seconds.

(*iii*) With continuous brakes the force that can be applied to stop the train is $\frac{1}{7} \times 135$ tons. If t be the time required in seconds, then, the impulse

$$= \frac{135}{7} \times t = \frac{135}{32.2} \times 66, \text{ (change in momentum)}$$

$$\text{or, } t = \frac{7 \times 66}{32.2} = 14.34 \text{ secs.}$$

Illus. Ex. 106. *With a jack-screw two men are lifting a load of 6 tons. The screw has two square-threads per in. of length; the mean diameter of the screw is 2.5 ins.; the coefficient of friction is .02; the length of the handle measured from the axis is 20 ins. Find the force exerted by each man.*

(*N.C.E. 1940*)

The pitch of the screw $p = \frac{1}{2}$ in.

If α be the inclination of the threads, $\tan \alpha = \dfrac{p}{\pi d}$.

$$= \frac{1}{2 \times 3.14 \times 2.5} = .0637.$$

But $\tan \lambda = .02$. If P be the force that should be produced at the mean diameter, $\frac{P}{6} = \tan(\alpha + \lambda)$ (Art. 207-II)

$$= \frac{\tan \alpha + \tan \lambda}{1 - \tan \alpha . \tan \lambda}$$

$$= \frac{.0637 + .02}{1 - (.0637 \times .02)} = \frac{.0837}{.9987} = 0838$$

Torque produced by the force P to rotate the screw is equal to the torque produced by the force applied by the two men.

Therefore, $.5028 \times 2240 \times 1.25 = 2 \times p \times 20$, where p is the force applied by each man in lbs.

$$\therefore \; p = \frac{.5028 \times 2240 \times 1.25}{2 \times 20} = 35.2 \text{ lbs.}$$

Illus. Ex. 107. *A rod 20 feet long and weighing 32 lbs. rests with its one end against a smooth vertical wall and the other end on a rough ground 12 feet from the foot of the wall. If the weight of the rod is assumed to act at the centre of its length vertically downwards, find the magnitude and direction of the forces exerted by the wall and the ground on the rod. Find also the co-efficient of friction between the rod and the rough ground.*

AB (Fig. 161) is the position of the rod. Weight 32 lbs. acts vertically downwards at C, the middle point of AB. Because the wall is smooth the reaction of the wall, F must be at right angles to it. The lines of action of these two forces meet at the point, (say), O. Join OB. Then BO is the line of action of the third force, the reaction of the ground Q, which keeps the system in equilibrium.

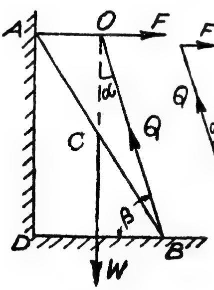


FIG. 161

Taking the moments of the three forces about B,

$$F \times AD = W \times \tfrac{1}{2} BD,$$

i.e., $F \sqrt{20^2 - 12^2} = 32 \times 6.$

$$\therefore \; F = \frac{32 \times 6}{16} = 12 \text{ lbs.}$$

From the vector triangle in the figure, $Q^2 = F^2 + W^2$

or, $Q = \sqrt{12^2 + 32^2} = 34.18$ lbs.

Again, $\tan \alpha = \frac{F}{W} = \frac{12}{32} = .375. \quad \therefore \; \alpha = 20° - 36'$

and $\beta = 69° - 24'$

If Q be resolved into two directions—vertical and horizontal, the vertical component is $Q \sin \beta = W = 32$ lbs., which is the normal reaction of the

ground. The horizontal component of Q is $Q \cos \beta = F = 12$ lbs. which is equal to the frictional force.

Therefore, $F = \mu . W$ or, $\mu = \frac{F}{W} = \frac{12}{32} = .375$

Illus. Ex. 108. *A uniform ladder of weight W, inclined to the horizon at 45°, rests with its upper extremity against a rough vertical wall and lower extremity on the ground. If μ and μ' be the coefficients of friction at the lower and upper ends respectively, show that the least horizontal force which will move the lower end towards the wall is just greater than*

$$\frac{W}{2} \cdot \frac{1 + 2\mu - \mu\mu'}{1 - \mu'}$$

Let P be the least force (Fig. 162) applied to attain the limiting condition of equilibrium. The upper end has a tendency to slide up the wall, whereas, the lower end has to move towards the wall. Therefore, the reactions on the

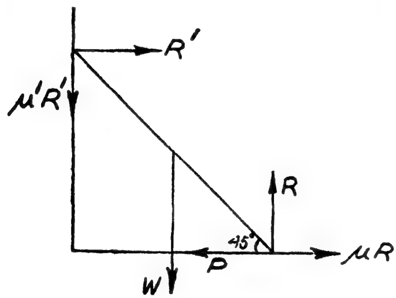


FIG. 162

upper end is R' and $\mu' R'$, where R' acts normally. Similarly, at the lower end the forces acting are R, P and μR — R being the normal pressure. The weight of the ladder will act at the centre of the length as the ladder is uniform.

Now, because the system of the forces is in equilibrium, the sum of the horizontal components,

$$R' + \mu R - P = 0 \qquad \ldots \qquad (i)$$

And also the sum of the vertical components,

$$W + \mu' R' - R = 0 \qquad \ldots \qquad (ii)$$

Again, taking the moments of all the forces about the lower end,

$W \times \frac{1}{2} l + \mu' R' l = R' \times l$, where l is the length of the portion of the ground concerned.

Therefore, $R' = \frac{W}{2(1 - \mu')}$

Substituting the value of R' in the equation (*ii*),

$$R = W + \mu' \times \frac{W}{2(1-\mu')} = \frac{W}{2}\left(2 + \frac{\mu'}{1-\mu'}\right)$$

Again, substituting the values of R and R' in the equation (*i*),

$$P = \frac{W}{2(1-\mu')} + \frac{W}{2}\left(2 + \frac{\mu'}{1-\mu'}\right)\mu = \frac{W}{2}\left\{\frac{1}{1-\mu} + \frac{2\mu - \mu\mu'}{1-\mu'}\right\}$$

$$= \frac{W}{2} \quad \frac{1 + 2\mu - \mu\mu'}{1-\mu'}$$

Therefore, to create motion in the ladder the force applied must be a bit greater than this value

Illus. Ex. 109. *A uniform ladder rests with one end on a horizontal stone-pavement, the other being leaning against a vertical brick-wall, prove that at the limiting position of equilibrium, the inclination θ of the ladder to the horizontal is given by,*

$$\tan\theta = \frac{1 - \mu\mu'}{2\mu}$$

where μ and μ' are coefficients of friction at the pavement and the wall respectively

Taking the idea from Fig 162 and adjusting the proper directions to the frictional forces,

$\Sigma H = R' - \mu R = 0$, where R' and R are the normal reactions of the wall and the pavement respectively (*i*)

And, $\Sigma V = \mu' R' + R - W = 0$, where W is the weight of the ladder, assumed to act at the middle of it due to its uniformity (*ii*)

Now, taking the moments of all the forces about the point where it touches the ground, $l \cos\theta \; \mu' R' + l \sin\theta \; R' - W \frac{l}{2} \cos\theta = 0$, where l is the length of the ladder Dividing both sides of the equation by $l \cos\theta$, and equating,

$$R' = \frac{W}{2(\mu' + \tan\theta)}$$, substituting this value of R' in (*ii*),

$$R = W - \frac{W\mu'}{2(\mu' + \tan\theta)}$$

Now, substituting the values of R and R' in (*i*) and multiplying both the sides of the equation by $2(\mu' + \tan\theta)$ and dividing by W,

$1 - 2\mu(\mu' + \tan\theta) + \mu\mu' = 0$ From which, $\tan\theta = \frac{1 - \mu\mu'}{2\mu}$

Illus. Ex. 110. *The weight of the crank shaft of an engine with its mountings is 3 tons. The average crank effort is 2000 lbs. The crank length is 10 ins. and the diameter of the shaft is 2.5 ins. If the coefficient of friction between the bearing surface and the shaft surface be .02 and if the shaft rotate with 300 r.p.m., find the Brake Horse Power of the engine neglecting the consideration of the resistances in other parts.*

Resistance due to friction in the bearing

$$= .02 \times 3 \times 2240 = 134.4 \text{ lbs}$$

Crank effort = 2000 lbs.

$$\therefore \; B.H.P. = \frac{2 \times \pi \times 300\,(2000 \times 10 - 134.4 \times 1.25)}{12 \times 33000}$$

$$= 94.34.$$

LUBRICATION

212. Sliding friction between non-lubricated and dry surfaces was being discussed so long. There, the three laws originally based on the experiments made by Morin and Coulomb were explained.

In the Mechanism of machines this frictional force generally acts in two different ways. Firstly, the motion and power is transmitted through friction, as in cases of couplings in the workshop line shafts, belts and rope drives, etc. Secondly, the force of friction acts against the transmission of power reducing the efficiency of machines, as in cases of journals and bearings. For the first group the question of lubrication does not arise, but in the latter cases proper lubrication is the prominent factor for machine efficiency. In these cases devices are made to reduce the value of μ, to increase the efficiency. Surfaces are made as smooth as possible and some kinds of lubricants are applied between the surfaces in contact. Common lubricants are oils of various kinds—olive oil, lard oil, rape oil, mineral oil, etc. There are different methods of lubrication. The two fundamental principles are,—(1) oil is allowed to flow under ordinary atmospheric pressure, and, (2) oil is forced under a pressure of 15 lbs. per sq. inch or upwards according to needs. The temperature of the bearing is not allowed to rise, ordinarilly above 100° F with full load constant running. On the first principle generally two methods are adopted. In one method, *Needle-lubricator* as shown in diagram (Fig. 163) is used. An oil-reservoir, as shown, is fitted on the bearing cap with the help of a wooden stopper. A needle, *P*, which is a piece of straight

wire, passes through a small passage through the wooden stopper and the brass. The upper end of the wire is flattened so that it cannot slip through the passage and rests in the oil-reservoir. The lower

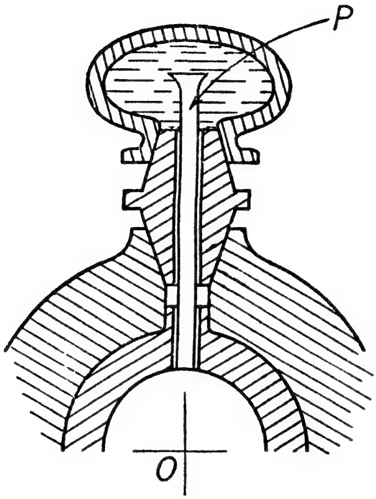


FIG. 163

end simply rests on the surface of the journal. Due to the capillary action oil does not travel through the passage, but when the journal is in motion, the vibration set up creates a gliding motion in the oil along the needle surface.

In the second method, a *syphon lubricator*, as shown in the diagram (Fig. 164), is used. In this system there is a reservoir made from the same casting with the cap of the bearing as shown. Oil slowly flows along the wick *C* due to the syphonic action. The upper end of the wick must remain below the free surface of the oil in the reservoir.

The second is generally adopted in cases where sufficient journal length for proper lubrication by ordinary method cannot be obtained. A pump mechanism is adjusted in the bearing to push the lubricating oil between the bearing and the journal surfaces where the normal pressure is maximum.

There are also other simpler methods of lubrication. As for example, pad lubrication, ring lubrication, bath lubrication, etc. In pad lubrication soft pad saturated with lubricant takes the place of

a portion of the bearing where there is no pressure. In ring lubrication, there is a ring or are rings suspended round the journal. The rings are bigger in size than the journal diameter. The bearing contains an oil bath in which a portion of the rings is submerged into the oil. When the journal rotates, due to friction the rings also roll over the surface of the journal and thus carrying the oil from the oil bath spread the same over the surface of the journal.

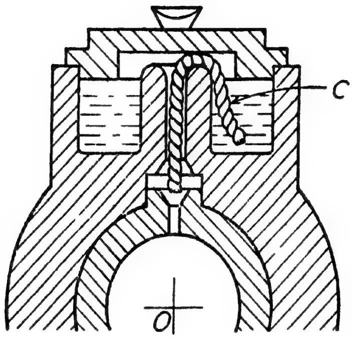


FIG. 164

In bath lubrication, a portion of the journal itself remains submerged in the oil bath in the bearing.

Whatever may be the method of lubrication the main object of using the lubricants is to reduce the frictional force so that the work done by this force against the motion of the machine, *i.e.*, the loss becomes less and the wearing is diminished, thereby increasing the life of the surfaces and saving money against repeated renewing of the machine parts. The lubrication may be intermittent as well as continuous.

In cases of smooth lubricated surfaces it is found that the force of friction follows more the laws of fluid friction than those of the solid friction. In cases of fluids the force of friction is found to be approximately,

(1) independent of the normal pressure,

(2) proportional to the area of surfaces in contact,

and (3) proportional to the square of the sliding velocity.

However, if we stick to the relation, $F = \mu R$, it is found on experiments that while the value of μ in case of dry smooth metal surfaces is 0.15 to 0.35, depending on the materials of the surfaces, in case of intermittent lubrication it falls up to a value 0.08 and in case of continuous lubrication even up to 0.03.

213. Tower's Experiments. Mr. Beauchamp Tower's experiments are the most exhaustive and authoritative in the line. On the results of the experiments he deduced a relation between μ, v and p, where μ is the coefficient of friction, v is the surface velocity of the shaft in feet per second and p is the intensity of pressure in pounds per square inch of the projected area of the bearing, $\mu = \dfrac{c\sqrt{v}}{p}$, c being a constant depending on the nature of the lubricant used. For rape oil it is 0.213, for mineral oil it is 0.27 and for olive oil it is 0.29 and for other oils values are different. For different methods of lubrication—syphon, bath and pad—at pressures varying from 252 to 272 pounds per sq. inch and at a velocity of 157 feet per minute he obtained the values of μ as 0.0098, 0.0014 and 0.009 respectively. The general equation, $\mu = \dfrac{c\sqrt{v}}{p}$, is used in case of bath lubrication for mineral oil.

Following his method it is found that in case of syphon lubrication with rape oil the relation, $\mu = \dfrac{2}{p}$ may be used at all speed and in case of pad lubrication with rape oil the value of μ may be taken as 0.01 at all speed.

214. Friction and Power Transmission. Friction between band and pulley surfaces :—

(*a*) *Belt drive* : Let the portion AB of the belt (Fig. 165) be in contact with the pulley surface. The angle of the grip is θ. Suppose the pulley rotates in the anti-clockwise direction. Due to the friction the tension T_1 in the tight side must be greater than the tension T_2 in the slack side. Consider an element of length CD in AB and let it subtend an angle $d\,\theta$ at the centre of the pulley. Then, if T be the tension at D, the tension at C will be $T + d\,T$. Let the normal pressure on CD due to the tightening of the belt be R. Because T and $T + d\,T$ are very nearly equal to each other, R may be said

to be approximately equal to $T \times d\,\theta$ (vector diagram in the Fig. 165-I).

$$T + d\,T - T = d\,T = \mu\,R = \mu\,T \times d\,\theta, \qquad \text{or,} \quad \frac{d\,T}{T} = \mu\,d\,\theta.$$

Now, integrating both the sides between the limits,

$$\int_{T_2}^{T_1} \frac{d\,T}{T} = \int_{0}^{\theta} \mu\,d\,\theta, \qquad \text{or,} \qquad \log_e T_1 - \log_e T_2 = \mu\,\theta,$$

$$\text{or,} \qquad \log_e \frac{T_1}{T_2} = \mu\,\theta \qquad \text{or,} \quad \frac{T_1}{T_2} = e^{\mu\,\theta}$$

where, e = Naperian base, 2.718.

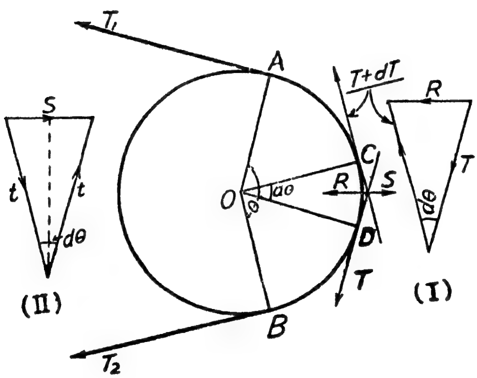


FIG. 165

Again, if common logarithms be used $\log \dfrac{T_1}{T_2} = \log e^{\mu\,\theta}$

$$= .4343\,\mu\,\theta.$$

$$\ldots\ldots Eq.\ 118$$

if the angle of grip be measured in degrees,

$$\log \frac{T_1}{T_2} = .4343 \times \mu \times \frac{\pi}{180}\,n.$$

$$= .007578\,\mu\,n$$

where, n is the angle in degrees.

Thus, it is evident that the friction is dependent on the angle of grip. Therefore, from this point of view, cross-be't connection is preferable in transmitting power.

Now, in deducing the above formula it is to be marked that the consideration of the effect of the centrifugal force acting on the belt has not been taken into account. This formula is quite sufficient for a moderate belt speed. But, when the speed is high the centrifugal force tends to diminish the normal pressure on the pulley. Therefore, the tension in the belt should be such that it includes both the transmitting force and the force required to neutralise the effect of the centrifugal force arising out from the circular motion of the belt over the pulley surface. Hence, in calculating the belt section both these forces must be considered.

If v be the speed of the belt in feet per second, w be the weight of the belt per unit length in pounds, the centrifugal force s on the elementary portion CD (Fig. 165) $= \frac{w}{g} r.\, d\theta.\ \frac{v^2}{r} = \frac{w}{g} v^2.\ d\theta$, where r is the radius of the pulley.

When the pulley rotates with uniform speed, this centrifugal force must be the resultant of two equal forces acting at C and D as tensions. If the magnitudes of the tensions be represented by t,

$$t \sin\frac{d\theta}{2} = \frac{1}{2}\frac{w}{g} v^2\, d\theta \qquad \text{(Fig. 165-II)}.$$

$$\text{or,} \quad t = \frac{w}{g} v^2 \times \frac{d\theta}{2 \sin\frac{d\theta}{2}}$$

And at the limit as $d\theta$ decreases, $t = \frac{w}{g} v^2$

Thus, when the centrifugal force is considered, the effective tensions on the tight and slack sides of the belt becomes $T_1 - t$ and $T_2 - t$ respectively. Hence,

$$\frac{T_1 - t}{T_2 - t} = e^{\mu\theta} \qquad \text{.........Eq. 119}$$

But the work done per second, $Q = (T_1 - t - T_2 + t)v = (T_1 - T_2)v$, and hence, the horse-power transmitted,

$$H.\ P. = \frac{(T_1 - T_2)v}{550} \qquad \text{............Eq. 120}$$

In determining the centrifugal force the weight of 12" × 1" double leather belt is taken as 0.15 lb. The value of μ in case of leather belt over iron is taken as 0.3 and that of T_1 as 150 lbs. per inch width of the belt.

35

By the term "high speed" is meant a speed over 2000 feet per minute. In dynamo driving the belt speed is allowed to attain 6000 to 7500 feet per minute.

(*b*) *Rope drive.* Belts are fitted on flat surfaces of pulleys, but ropes are fitted on pulleys, the surfaces of which have V-groove as shown in the diagram (Fig. 166). The rope is in contact at two surfaces. The normal pressure on each of the surfaces, $R_1 = \frac{1}{2} R \div \sin \frac{\alpha}{2}$

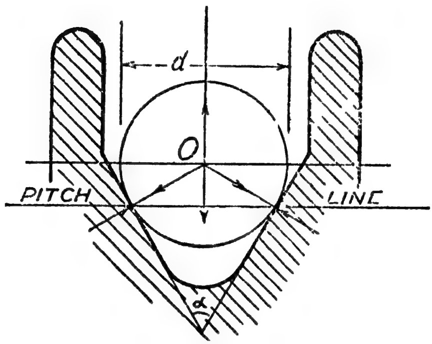


FIG. 166

i.e., more than half the pressure, R, that would fall if the rope was fitted on flat surface. Here, α is the angle of the groove. Now,

the resistance to slipping $= 2\,\mu\, R_1 = 2\,\mu \times \dfrac{\frac{1}{2}R}{\sin\frac{\alpha}{2}} = \mu R \operatorname{cosec} \dfrac{\alpha}{2}$.

Thus, if in the relation between the frictional force and the normal pressure in case of a belt, $F = \mu\, R$, $\mu \operatorname{cosec} \frac{\alpha}{2}$ be substituted for μ, then, the relation between the resistance to slipping of the rope and the resultant of the normal reactions of the grooved surface is obtained. Hence, if $k = \operatorname{cosec} \frac{\alpha}{2}$, where k is a constant (depending on the angle of the groove), $\dfrac{T_1}{T_2} = e^{k\,\mu\,\theta}$. ....*Eq.* 121

Generally the angle of the groove is made 45°; in that case $\dfrac{T_1}{T_2} = e^{2.613\,\mu\,\theta}$, and allowing for the effect of the centrifugal

force the equation will be $\frac{T_1 - t}{T_2 - t} = e^{2.613 \mu \theta}$, where $t = \frac{w}{g} v^2$ (w being the weight of the rope per foot length and v being the velocity of the rope in feet per second).

The emperical relation for assuming the weight of Manila hemp rope (which is generally used) per foot length is, $w = 0.34\, d^2$ lbs., where d is the diameter of the rope in inches. The value of μ between the rope and the iron-grooved surface is taken as 0.13.

Illus. Ex. 111. *A Manila rope $1\frac{1}{4}$ inches in diameter is wound round a drum 3 times. When the tight side reaches the limit of the breaking strength, what should be the tension in the slack side? The breaking strength of the rope is 12000 lbs. and $\mu = .4$.*

$$\log \frac{12000}{T_2} = .4 \times 2\pi \times 3 \times .4343$$

$\log T_2 = \log 12000 - 3.274 = 4.0792 - 3.274 = .8052$

Therefore, T_2, the tension in the slack side $= 6.386$ lbs.

Illus. Ex. 112. *The belt speed of a certain machine is 5000 feet per minute to transmit 50 horse power. If the tension in the tight side is 180 lbs. per inch width of the belt, the angle of grip is 150° and $\mu = .25$, find the width of the belt. The weight of the belt 1 foot long and 1 inch wide is .15 lbs.*

$$\text{Centrifugal tension, } t = \frac{w}{g} v^2 = \frac{.15}{32.2}\left(\frac{5000}{60}\right)^2$$
$$= 32.3 \text{ lbs.}$$

$$\log \frac{180 - 32.3}{T_2 - 32.3} = .4343 \times .25 \times \frac{150}{180} \times 3.14 = .2842$$

$\log 147.7 - .2842 = \log (T_2 - 32.3)$

or, $\log (T_2 - 32.3) = 2.1693 - .2842 = 1.8851$

or, $T_2 - 32.3 = 76.76$. $\therefore$ $T_2 = 109$ lbs.

Per inch width of the belt, the effective pull,

$T_1 - T_2 = 180 - 109 = 71$ lbs.

Again, the total effective pull for transmitting 50 H.P. is found out as follows :

$$\frac{(T_1 - T_2) \times 5000}{33000} = 50$$

or, $T_1 - T_2 = 330$ lbs.

Therefore, the width of the belt required is $330 \div 71 = 4.7$ inches, *i.e.*, 5 inches belt should be used.

FRICTION AS RESISTANCE

215. Friction between journal and bearing. This friction may be classified into two different types—(1) Axle Friction, and (2) Pivot and Collar Friction.

216. Axle Friction.

(*a*) *When the journal is fitted horizontally in a bearing, assuming the materials of the elements to be incompressible theoretically and the surfaces of the journal and the bearing to be perfectly cylindrical, the two surfaces will be in contact in a single straight line.*

Let r be the radius of the shaft, which loosely fits in the bearing as shown in the diagram (Fig. 167—drawn, of course, in an exaggerated scale). Let the weight of the shaft with its mountings produce a resultant vertically downward. Now, if the shaft begins to rotate in the direction shown by the arrow-head, the shaft will first tend to roll up the bearing surface, and then with some line, as line of contact, whose trace is C (say), it will go on rotating. Before the shaft starts for actual rotation there are three forces in equilibrium—(1) Normal reaction, R, (2) Frictional force, F, and (3) combined effect of the pulls of the belts connecting the machines and the weight of the shaft with all its mountings, which is nothing but equal and opposite to the resultant of the former two, and call it W. Then, $W^2 = R^2 + F^2$, or, $W = \sqrt{R^2 + F^2}$.

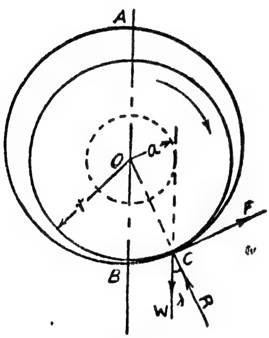


FIG. 167

The moment of W about O, the centre of the shaft section, must be equal to the sum of the moments of the components, F and R, about the same point. That is, $W \times a = R \times 0 + F \times r$, where a is the perpendicular distance of the line of action of the force W from O.

Therefore, $W.a = F.\ r = T$ (torque produced by the frictional resistance).

But, $F = \mu R$, and $\mu = \tan\lambda$. Hence, $F = R \tan\lambda = W \sin\lambda$.

Again, $\frac{a}{r} = \sin\lambda$, or, $a = r \sin\lambda$. In cases of properly lubricated surfaces the value of μ being very small, $\sin\lambda = \tan\lambda$

Therefore, $F = W \sin\lambda = W \tan\lambda = \mu W$,

$$\text{and } a = r\sin\lambda = r\tan\lambda = \mu r. \qquad \cdots\cdots\cdots Eq.\ 122$$

Also, $T = \mu W r$

If the speed of the shaft be N revolutions per minute, the loss of energy per minute due to friction $= 2\pi T N = 2\pi \mu W r N$.

Now, if with O as centre and a as radius a circle is drawn, then, the line of action of the resultant of F and R, *i.e.*, W in the opposite sense, will be tangent to the circle. This circle is called the *Friction Circle*. It is to be marked that the line of action of the resultant becomes tangent to the friction circle on the side towards which the journal has the tendency to roll.

(*b*) *If the shaft and bearing have a surface of contact.*

Let AB be the length of the surface of contact (Fig. 168), and let the diameter COD bisect it into two equal halves. Let the angles subtended by AD and DB at the centre be equal to α. If w be the intensity of pressure on the bearing surface (which is assumed to be uniform throughout the surface, which is, of course scarcely the case due to various reasons), then, the normal pressure on an elementary surface subtending an angle $d\theta$ at an angular distance of θ from the radial line OD is equal to $w.r.d\theta.l$, where l is the length of the bearing. Its component along $OD = w.r.d\theta.l \cos\theta$, and the resultant of all such components,

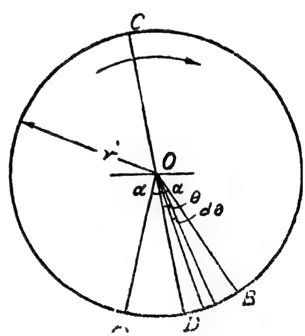


FIG. 168

$$W = \int_{-\alpha}^{+\alpha} w.r.d\theta\ l \cos\theta$$

$$= 2wrl\sin\alpha$$

Therefore, $w = \dfrac{W}{2\,r\,l \sin \alpha}$ ....*Eq.* 123

The resultant of all the components at right angles to *OD* and on both sides of it is Zero.

Again, the frictional force on the elementary area $= \mu\, w\, r\, l\, d\theta$ and the torque produced by the force $= \mu\, w\, r^2\, l\, d\theta$. Therefore, the total torque produced by the total frictional resistance on the surface of contact, $T = \int_{-\alpha}^{+\alpha} \mu\, w\, r^2\, l\, d\theta = 2\,\mu\, w\, r^2\, l\, \alpha$. Now, substituting the value of w in the equation above,

$$T = 2\,\mu\, r^2\, l\, \alpha \times \frac{W}{2\,r\,l \sin \alpha}$$

$$= \mu\, W r\, \frac{\alpha}{\sin \alpha} \qquad \cdots\cdots\ \textit{Eq. } 124$$

If the surface of contact be small, *i.e.*, if α be very small, $\alpha = \sin \alpha$, and $T = \mu\, W r$. That is, the line of contact and the surface of contact gives the same result with the assumption that the surface of contact is small. If the surface of contact subtends an angle of 60°, then $\alpha = 30°$ and the error is only 5 per cent.

It is to be noted that the intensity of bearing pressure in these cases was found by dividing the resultant pressure by the area obtained by the product of the length of the bearing and the length of the arc of contact. But, in general the intensity of bearing pressure in engineering computations is always represented by the pressure per unit of the diametral area to determine the bearing dimensions so that the shaft can run smoothly without producing much heat. The allowable pressure intensity is determined on experiments.

(*c*) *Axle friction in a* V-*grooved guide or bearing.*

Taking the contact as line contact, there are two lines of contact at *A* and *B* (Fig. 169). The lines of action of the reactions at the points of contact which are the resultants of the normal reactions and the frictional resistances, will be tangents to the friction circle. Let them meet at the point *K*. Then, their resultant will act through *K* and will be parallel and equal to the pressure *W* due to the load which includes the weight of the shaft and the driving forces, all of them

being assumed to act vertically downwards. The moment of this resultant about O, the centre of the axle section and that of the fric-

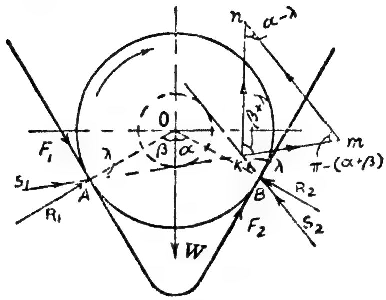


FIG 169

tion circle, will be equal to the sum of the moments of the frictional resistances F_1 and F_2. From the vector triangle, $k\,n\,m$,

$$\frac{W}{\sin\{\pi - (\alpha + \beta)\}} = \frac{R_1}{\sin(\alpha - \lambda)}, \quad \text{or,} \quad R_1 = W \frac{\sin(\alpha - \lambda)}{\sin(\alpha + \beta)}$$

$$\text{Similarly} \quad R_2 = W \frac{\sin(\beta + \lambda)}{\sin(\alpha + \beta)}$$

$$\text{Therefore,} \quad F_1 = W \frac{\sin(\alpha - \lambda)}{\sin(\alpha + \beta)} \sin\lambda$$

$$\text{and} \quad F_2 = W \frac{\sin(\beta + \lambda)}{\sin(\alpha + \beta)} \sin\lambda$$

$$\text{Hence,} \quad T = F_1 r + F_2 r$$

$$= \frac{Wr \sin\lambda}{\sin(\alpha + \beta)} \left\{\sin(\alpha - \lambda) + \sin(\beta + \lambda)\right\}$$

In case where the direction of W is vertically downwards, $\alpha = \beta$

$$\text{and,} \quad T = \frac{W\,r \sin\lambda}{\sin 2\alpha}(2 \sin\alpha \cos\lambda) = \frac{W\,r \sin\lambda \cos\lambda}{\cos\alpha}$$

$$= \frac{W\,a}{\cos\alpha} \cos\lambda \qquad \cdots\cdots\cdots\cdots Eq.\ 125$$

Now, $\dfrac{W\,r \sin\lambda \cos\lambda}{\cos\alpha}$ may again be put in the form,

$$\frac{W\,r \sin\lambda \cos\lambda \dfrac{\sin\lambda}{\cos^2\lambda}}{\cos\alpha \dfrac{\sin\lambda}{\cos^2\lambda}} = \frac{W\,r \tan\lambda}{\cos\alpha \dfrac{1}{\cos^2\lambda}} = \frac{W\,r \tan\lambda}{\cos\alpha \dfrac{\sin^2\lambda + \cos^2\lambda}{\cos^2\lambda}}$$

$$\therefore\ T = \mu \frac{W\,r}{\cos\alpha\,(1 + \mu^2)} \qquad \cdots\cdots\cdots Eq.\ 126$$

217. Pivot and Collar friction.

When a shaft or axle has got a thrust along its axis, as in case of a vertical or inclined shaft in a workshop, the main spindle of a lathe, the propeller shaft in a ship, etc. the end-thrust is taken up by a support or bearing against which the shaft rotates. In some cases the end of the shaft is in direct contact with the bearing surface when the end of the shaft acts as a pivot, and in some cases collars are made in the shaft to stand against the end-thrust. The collars may be made at the end as well as at any portion of the shaft length.

The distribution of the thrust is assumed to be the same in all kinds of pivot and collar bearings, whether the shaft is a vertical one or a horizontal or an inclined one. First, let us consider the cases of pivot friction.

Pivots—The coefficient of friction is assumed constant. The cases of pivot friction are considered on two different assumptions : (1) the thrust is uniformly distributed, and (2) the wearing of the rubbing surfaces is uniform, being directly proportional to the product of the intensity of pressure and the velocity of sliding.

(*a*) PIVOTS WITH FLAT SURFACE

(1) *Uniformly distributed thrust* : If r_1 be the radius of the shaft section, the area $= \pi r_1^2$. Where W is the total thrust, the intensity of thrust, $w = \dfrac{W}{\pi r_1^2}$. If a ring area of width dr (Fig. 170) is considered at a radial distance r from the axis, the pressure on the ring area is equal to $2\pi r\,dr\,w$, and the frictional force

on it is $2\mu\pi r\,dr\,w$. The moment of this force about O, the centre of the section, is equal to $2\mu\pi r^2\,dr\,w$. Therefore, the total moment due to the total thrust W, $T = \int_0^{r_1} 2\mu\pi r^2\,dr\,w$

$$= \frac{2}{3}\mu\pi r_1^3 w = \frac{2}{3}\mu W r_1 \qquad Eq\ 127$$

(2) *Uniform wear* The wear at any point in the direction of pressure is proportional to the product of the intensity of pressure and the velocity of sliding. Therefore, if the wear is uniform the product of the intensity of pressure and the velocity may be said to be constant, or in other words, the product of the intensity of pressure and radius is constant, as the velocity at any point in the section is proportional to its distance from the axis of rotation. Hence, $w r = c$, where c is a constant.

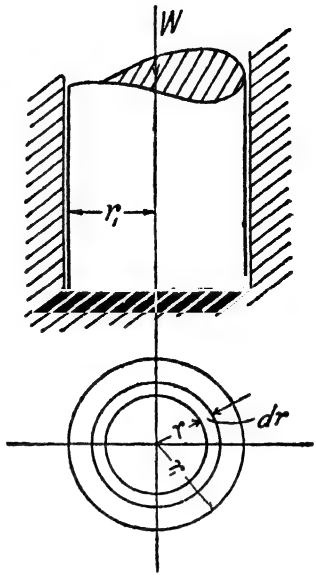


FIG 170

Now, the pressure on the ring area $= 2\pi r\,dr\,w = 2\pi c\,dr$. Therefore, the total pressure, $W = \int_0^{r_1} 2\pi c\,dr = 2\pi c r_1$, from which, $c = \dfrac{W}{2\pi r_1}$. Now, the frictional force on the ring area $= 2\mu\pi c\,dr$, and its moment about $O = 2\mu\pi c\,dr\,r$. Hence, the moment of the total frictional force on the pivot surface,

$$T = \int_0^{r_1} 2\mu\pi c\,dr\,r = \mu\pi c r_1^2$$

$$= \mu\pi r_1^2 \frac{W}{2\pi r_1} = \tfrac{1}{2}\mu W r_1 \qquad Eq\ 128$$

(b) PIVOTS WITH CONICAL SURFACE

(1) *Uniform pressure intensity* Let w be the intensity of normal pressure. Component of w parallel to the axis $= w\sin\theta$ (Fig 171), where θ is half the cone angle. The resultant of all these components $= w\sin\theta \times \dfrac{\pi r_1^2}{\sin\theta}$ (area of the cone surface) $= \pi w r_1^2$. The resultant of the horizontal components is zero, one being neutralised by another symmetrically situated on the other side of the axis.

Therefore, $W = \pi w r_1^2$, or, $w = \dfrac{W}{\pi r_1^2}$ — the form of the equation is just similar to that of the flat-surfaced pivot, and it can be said that the normal pressure on the cone surface is independent of the cone angle.

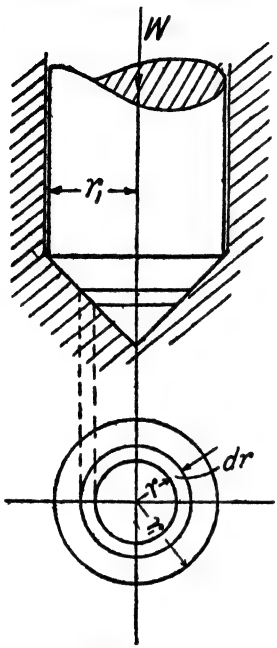


FIG. 171

Now, the normal pressure on the ring area $= \dfrac{2\pi r\, dr\, w}{\sin\theta}$ and the frictional resistance due to this pressure $= \dfrac{2\mu\pi r\, dr\, w}{\sin\theta}$. The moment of this force about $O = \dfrac{2\mu\pi r^2\, dr\, w}{\sin\theta}$

Therefore, the total moment due to the frictional force on the whole surface,

$$T = \int_0^t \frac{2\mu\pi r^2\, dr\, w}{\sin\theta}$$

$$= \frac{2\mu\pi w}{\sin\theta}\int_0^{r_1} r^2\, dr = \frac{2}{3}\,\frac{\mu W r_1}{\sin\theta}$$

$$\cdots\cdots\cdots Eq.\ 129$$

(2) *Uniform wear.* Proceeding in the same way as we did in cases of flat-surfaced pivot and with the help of the previous proof,

$$T = \frac{1}{2}\cdot\frac{\mu W r_1}{\sin\theta}. \qquad \cdots\cdots\cdots\cdots Eq.\ 130$$

(c) HOLLOW SHAFT END AS PIVOT

The treatment with a hollow shaft is the same with the shafts with collar which will be evident from the following discussions with collars.

218. Collars and Bearings. The assumptions made are the same with those of the pivot cases.

The diagram (Fig. 172) is of a hollow shaft, and the diagrams

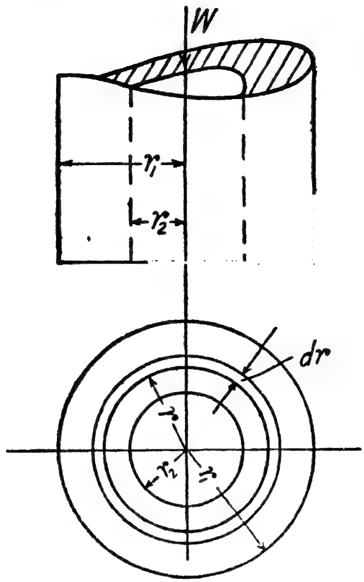


FIG 172

(Figs. 173 and 174) are of shafts with flat collars In the case where bearing surface required is big, multiple collars as in Fig. 174 instead of a single collar as in Fig 173 are used.

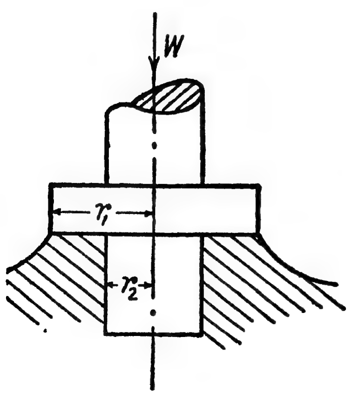


FIG. 173

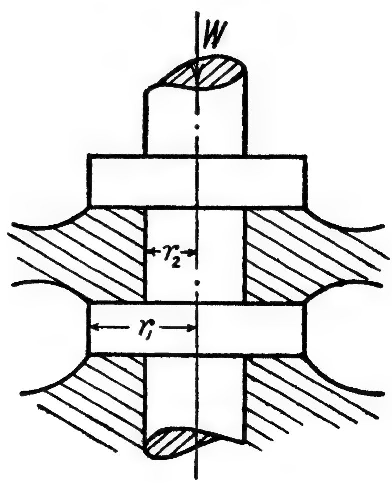


FIG. 174

(1) *Uniform pressure intensity.* Moment of the frictional resistance on the ring area

$$= 2\,\mu\,\pi\,r^2\,dr\,w,$$

r is the radial distance of the area.

$$\therefore\quad T = 2\,\mu\,\pi\,w \int_{r_2}^{r_1} r^2\,dr,$$

r_1 and r_2 are the radii of the collar section.

$$= 2\,\mu\,\pi\,w\,\frac{r_1^3 - r_2^3}{3}$$

Again, $w = \dfrac{W}{\pi\,(r_1^2 - r_2^2)}$.

$$\therefore\quad T = 2\,\mu\,\pi\,\frac{W}{\pi\,(r_1^2 - r_2^2)} \times \frac{r_1^3 - r_2^3}{3}$$

$$= \frac{2}{3}\,\frac{\mu\,W\,(r_1^3 - r_2^3)}{(r_1^2 - r_2^2)} \qquad \cdots\cdots\ Eq.\ 131$$

(2) *Uniform wear.*

Pressure on the ring area $= 2\,\pi\,r\,dr\,w = 2\,\pi\,c\,dr$.

Total load, $W = \displaystyle\int_{r_2}^{r_1} 2\,\pi\,c\,dr = 2\,\pi\,c\,(r_1 - r_2)$.

$$\therefore\quad c = \frac{W}{2\,\pi\,(r_1 - r_2)}.$$

Moment of the frictional resistance on the ring area $= 2\,\mu\,\pi\,c\,r\,dr$

$$\therefore\quad T = \int_{r_2}^{r_1} 2\,\mu\,\pi\,c\,r\,dr = 2\,\mu\,\pi\,c\,\frac{r_1^2 - r_2^2}{2}$$

$$= 2\,\mu\,\pi\,\frac{W}{2\,\pi\,(r_1 - r_2)} \times \frac{r_1^2 - r_2^2}{2}$$

$$= \tfrac{1}{2}\,\mu\,W\,(r_1 + r_2) \qquad \cdots\cdots\cdots\cdots\ Eq.\ 132$$

219. Schiele's pivot or Anti-friction pivot. The bearing surface is so designed that both the wear in the direction of the axis and the intensity, w, normal to the surface are uniform. The coefficient of friction is assumed to be constant. AC represents the vertical wear at A,

(Fig. 175), a point on the surface where the radius is r. AC is resolved into two components, AB and BC, perpendicular and parallel to AD respectively. AD is the length of the tangent to the curve at A, from the point of tangency to the axis of the pivot.

The wear AB is assumed to be proportional to the intensity of pressure w and the velocity of rubbing, *i.e.*, r (the velocity is proportional to r). Therefore, $AB \propto r$, *i.e.*, $AB = c \times w \times r$, where c is a constant quantity.

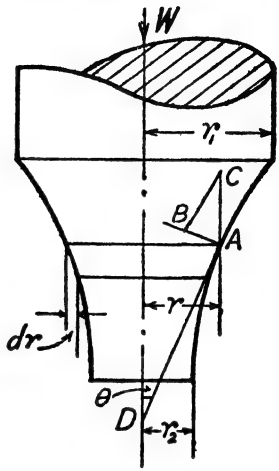


FIG. 175

From similarity of triangles, $\dfrac{AC}{AB} = \dfrac{AD}{r}$

or, $AC = AB \times \dfrac{AD}{r}$

But, $AC = c \times w \times AD$.

Hence, to retain the value of AC constant AD must be constant.

The moment of the friction on the ring surface of width dr at the radius r, is equal to $2\pi l w r\, dr$. Where $l = AD$ and the ring area

$$= 2\pi r \frac{dr}{\sin\theta} = 2\pi r l\, dr.$$

Therefore, the moment of resistance due to friction on the whole area,

$$T = 2\pi l w \int_{r_2}^{r_1} r \times dr = \pi l w (r_1^2 - r_2^2) \quad \cdots\cdots\cdots Eq. 133$$

$$\text{Again, } W = 2\pi w \int_{r_2}^{r_1} r \times dr = \pi w (r_1^2 - r_2^2)$$

$$\therefore \quad T = W l \quad \cdots\cdots\cdots Eq. 134$$

Thus, we find that the value of T, being proportional to l can be reduced by reducing the value of l. It is customary to take the value of l as r_1. Hence, $T = W r_1$ ············*Eq.* 135

Now, comparing the values of T in different kinds of pivots, it is clear that friction in a Schiele pivot is the greatest. But, the designer claims that the wear and pressure intensity being uniform the surfaces

will always fit well and the lubricant will not be squeezed out and run smoothly at high speeds.

The curve, showing the shape of the surface of the pivot, which is drawn so that its tangent AD is of constant length always, is called the *Tractrix* and also it is known as the *Anti-friction curve.*

Illus. Ex. 113. *A flat Pivot is to carry a load of 5024 lbs. .The intensity of pressure is taken to be uniform and is equal to 100 lbs. per square inch. Compute the horse power absorbed by the friction of the pivot if it runs at 150 r.p.m. Take* $\mu = .006$.

The cross-sectional area of the shaft $= \dfrac{5024}{100} = 50.24$ sq. ins.

Therefore, $50.24 = \pi r^2$, and $r = 4$ inches.

Torque $= \frac{2}{3} \times .006 \times 5024 \times 4 = 80.38$ lb. ins. (*Eq.* 127)

Hence, the horse power absorbed $= \dfrac{2\pi \times 80.38 \times 150}{12 \times 33000}$

$= .1913$

ROLLING FRICTION

220. In case of cylindrical bodies, such as, rollers, wheels, etc. resting on a plane surface, the rolling motion is created due to a couple produced by the pulling force at the axle and an equal amount of sliding friction in the opposite direction at the surface of contact between the body and the surface. The resistance to rolling motion, which is named as rolling friction, is much less than the sliding friction in a sliding body. That is, much less force is required to create a rolling motion than to create a sliding motion. The rolling friction is measured by the relation already established in case of sliding friction, $F = \mu R$, but in this case μ represents coefficient of rolling friction and not of sliding friction. Because it is found that the quantity of force required to create a rolling motion is much less than what is required for a sliding motion, the value of the coefficient of rolling friction is much less than the coefficient of sliding friction. The friction in cases of rolling bodies is considered and treated in a different way as follows :

When a body rolls on a surface, its material is compressed and the surface of contact, bc (Fig. 176) takes the form, something like as shown in the diagram (drawn in an exaggerated scale). Let W be

the weight of the roller and F be the least force required to set the body in rolling motion. The resultant of W and F, which is equal and opposite to the reaction of the surface on the body, will pass through some point, (say) c in the surface of contact. Then, taking the moments of all the three forces acting in the body about c, $F.\,Od = W.\,cd$. But Od is very nearly equal to r, the radius of the body. Therefore, $F \times r = W \times a$ (very nearly), where a is the perpendicular distance of the point c from the vertical diameter. Thus, the torque of the pulling force is equal to $W.\,a$. Before the body takes

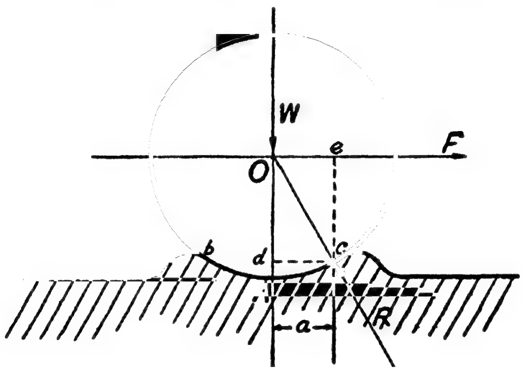


FIG. 176

a start the whole system of force is in equilibrium, *i.e.*, $\Sigma H = 0$. Therefore, the pulling force is equal to the rolling resistance. Thus, the moment of the rolling resistance about the centre is equal to $W \times a$. Compare this problem with that of art. 216 (a). In that case too the shaft was considered to roll before it took start to rotate. In both the cases the pulling forces and the radii being equal to F and r respectively, the torques due to the pulling forces are equal to the same value, $F \times r$, which is again equated to $W \times a$ to maintain the condition of equilibrium. Therefore, the value of a in both the cases is the same.

From the relation, $F \times r = W \times a$,

$$\text{we get, } F = \frac{a}{r} W \qquad \cdots\cdots\cdots\cdots \textit{Eq. } 136$$

It is to be marked that this article does nothing but contain the elaborate discussion of the actual condition of the surfaces of contact in article 216 (a) and the relation between the load and the effort.

In sliding friction, F (force of friction) was found to depend on the value of μ (the coefficient of sliding friction), but here, in this case, F, the least rolling force, *i.e.*, the rolling friction depends on the value of a — the radius remaining constant. In case of rolling bodies this distance a, measured in inch unit, is expressed as the coefficient of rolling friction. According to some experimentalists the value of a does not change when r varies, but some others say that the value changes with the radius of the body. It is evident that the rolling friction, *i.e.*, the value of a increases with the softness of the materials of the body and the surface, and it becomes less and less as the materials become harder and harder.

In case of sliding friction, $\frac{F}{W} = \mu$, and in case of rolling friction, $\frac{F}{W} = \frac{a}{r}$. Therefore, the form of the relation between F and W in sliding friction may be utilised in the computations of the problems on rolling friction if the value of μ in the equation of the sliding friction is taken as $\frac{a}{r}$.

From the experiments of the different experimentalists, like Coulomb, Pambour, Rittinger, etc., it is found that the value of a varies from 0.0183 inch to 0.0216 inch according to the materials used.

221. Tables for the value of μ and a.

Materials	μ	a
Wood on Wood	.3 to .7	.019 to .032
Cast iron on Cast iron	...	.0185 to .0195
Iron on Iron	.1 to .3	.015 to .021
Iron on Wood	.2 to .5	.5 to .1
Leather on Iron	.3 to .6	...

Illus. Ex. 114. *A railway wagon with load weighs 50000 lbs. Compute the pull maintained at the axle when the train runs with constant velocity. The axle is 4 inches in diameter and the diameter of the wheel is 36 inches. The coefficient of sliding friction is .08 and the coefficient of rolling friction is .015 inch.*

The force system acting in the wagon is transferred to the plane of a single wheel and the problem is solved.

$$\Sigma H = 0$$

Therefore, the pulling force must be equal to the horizontal component of the reaction at the point of contact.

$$\text{Again,} \quad \Sigma V = 0$$

Therefore, the weight is equal to the vertical component of the reaction.

Now, taking moments of all the forces in the system about the axis of the wheel,

$$P \times R \text{ (very approximately)} = W \times a + \mu W \times r + \mu P \times r$$

Let $\mu r = a_1$ (radius of the friction circle in case of journal and bearing).

$$\text{Then,} \quad PR = W(a + a_1) + Pa_1$$

$$\text{or,} \quad P(R - a_1) = W(a + a_1), \qquad \text{or,} \qquad P = \frac{W(a + a_1)}{R - a_1}$$

But a_1 is so very small in comparison with R that the consideration of its value can be neglected when added to or subtracted from R.

$$\text{Therefore, } P \text{ (very nearly)} = \frac{W(a + a_1)}{R}$$

Now, substituting the numerical values,

$$P = \frac{50000\,(.015 + .08 \times 2)}{18} = 486 \text{ lbs.}$$

Here it is to be marked that practically the value of $\mu P \times r$ has been neglected. Hence, the problem can be solved in an alternative method as follows :

In the Fig. 177, as the axle rotates in the clockwise direction, it will try to roll towards the left and hence the line of action of W will be tangent to the friction circle as shown. Then, the three forces namely W, P and the reaction at the point of contact of the wheel with the track, are in equilibrium. Now, taking the moments of all the three forces about the point of contact the same relation with the previous method is obtained.

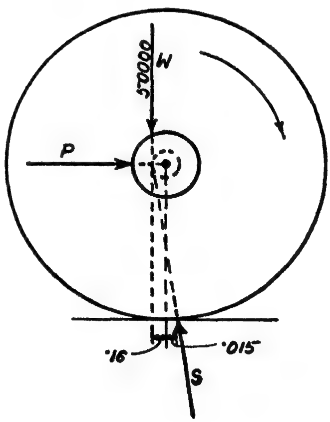


FIG. 177

222. Force required to drag a body on rollers :

Let W be the weight of the body, w be the weight of the roller, r be the radius of the rollers and a_1 and a_2 be the coefficients of rolling friction between the body and the roller, and the roller and the surface respectively (Fig. 178).

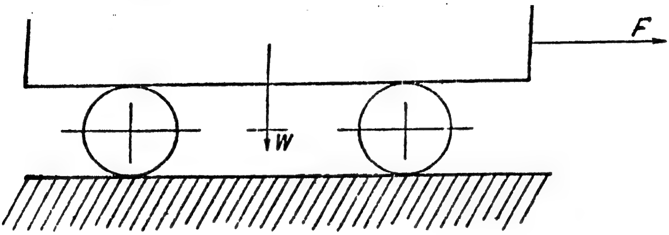


FIG. 178

The speed of the body = the peripherial speed of the roller + the speed of the axles, *i.e.*, = twice the speed of the axles.

Now, if F be the least force to drag the body, then, by the principle of work, the work done by F must be equal to the work done by the force applied at the axles of the rollers. Therefore, F is equal to half the force applied at the axles, which is equal to

$$\frac{W}{r} a_1 + \frac{W + w}{r} a_2$$

$$F = \tfrac{1}{2}\left(\frac{W}{r} a_1 + \frac{W + w}{r} a_2\right)$$

$$= \tfrac{1}{2}\left\{\frac{W}{r}(a_1 + a_2) + \frac{w}{r} a_2\right\}$$

Where $a_1 = a_2 = a$

$$F = \frac{a}{r}\left(W + \frac{w}{2}\right) \qquad \cdots\cdots\cdots\cdots Eq.\ 137$$

In case where the weight of the roller is negligible in comparison with the load pulled,

$$F = \frac{a}{r} W \text{ (approximately).} \qquad \cdots\cdots\cdots\cdots Eq.\ 138$$

It is to be marked that the result is the same with a roller of weight W being dragged on a horizontal track with a pull F at the axle.

223. Roller and Ball Bearing. In ordinary journal and bearing there is a sliding friction between the surfaces of the journal and the bearing. As the rolling friction is much less than the sliding friction, a special kind of bearings—roller and ball bearings—are used, so that, instead of a sliding friction between the two surfaces in contact, a rolling friction between them is created and the loss of work done is reduced thereby.

Balls or rollers are fitted between the journal surface and the bearing surface. A diagram for ball bearing is shown in Fig. 179. The shaft and the bearing surfaces are not directly in contact with the ball surfaces. One collar A on the shaft, and one bush B in the

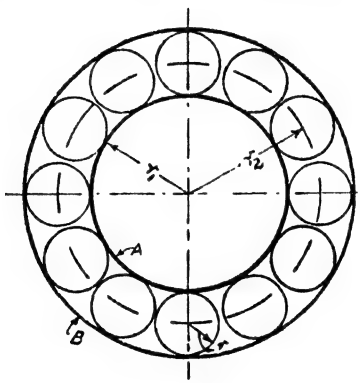


FIG. 179

bearing, which are called inner and outer races respectively are fitted. The first one is tight press fit, so that there is no creep between the shaft and the race, and thereby preventing wear which may create slackness of fit. The second one is only press fit with the bearing case, the stationary element of the turning pair. In this case a small amount of creep is beneficial, because it allows every portion of the inner surface of the outer race to be in contact with the ball surfaces at the loaded position and thereby undergoes uniform wear. The balls are arranged between the surfaces of the races as shown. There are grooved tracks on the contact surfaces of the races. The balls are usually separated by a metal cage, so that they remain at a constant distance without being crowded, otherwise, the efficiency of rolling is

destroyed. According to the length of the journal the number of rows of the balls may be more than one. The metal cage is something like ring plate having holes for the balls to be loosly fitted.

The balls, rollers and the races are made of very hard metal, best quality of Chromium Steel with high percentage of carbon is used. For low speed the cage metal is pressed steel and for high speed it is made of Bronze.

The difference between a roller bearing and a ball bearing is that there having a grooved track for the balls in ball bearing, it can sustain an amount of end thrust, whereas, in case of roller bearing the surfaces of the races are completely cylindrical and cannot sustain any end thrust.

The types of ball and roller bearings are shown in Figs. 180, 181 and 182.

FIG. 180

FIG. 181

FIG. 182

Fig. 180 is a view of the bearing with its balls, races and case shown separately. Fig. 181 is the view of a ball bearing and Fig. 182 is the view of a roller bearing.

Illus. Ex. 115. *A shaft in a roller bearing is 3 inches in diameter. The rollers are ¼ inch in diameter. The pressure on the bearing is 10000 lbs. Compute the loss of work per revolution due to the friction in the bearing. The coefficient of rolling friction is .02 inch.*

(1) Assume that the entire pressure is on one roller only; (2) assume that the entire pressure is distributed uniformly over half of the bearing surface.

(1) Neglecting the weight of the roller, the frictional

$$\text{resistance} = \frac{W}{r}\,(a + a_1) \qquad \text{(Art. 222)}$$

Its moment about the axis $= \dfrac{W}{r}(a + a_1) \times$ radius of the roller centre circle.

But, $a = a_1$

$$\therefore \text{ the moment} = \frac{2\,W\,a\,r_2}{r_1} \qquad \text{(Fig. 179)}$$

$$\text{Hence, work per revolution} = \frac{2\,W\,a\,r_2}{r_1} \times 2\,\pi$$

$$\text{Substituting the numerical values, } Q = \frac{2 \times 10000 \times .02 \times 1.625 \times 2 \times 3.14}{1.5 \times 12}$$

$$= 226{,}8 \text{ ft. lbs.}$$

(2) From *Eq.* 123 of article 216, the intensity of pressure is given as,

$$w = \frac{W}{2\,r\,l \sin \alpha}\text{, where } w \text{ is the intensity per unit area.}$$

If the intensity is expressed in lbs. per unit length of the area, *i.e.*, neglecting l in the equation,

$$w = \frac{W}{2\, r \sin \alpha}. \qquad \text{Here, } \alpha = 90°$$

$$\therefore \quad w = \frac{W}{2\, r}$$

When the rollers are near together, the intensity can be taken as $\frac{W}{2\, r_1}$.

If the number of the rollers be n and the pressure on each roller be p, then

$$\text{the total pressure} = n\, p = \frac{W}{2\, r_1}\, \pi\, r_1 = \frac{W\, \pi}{2}$$

$$\text{The moment} = \frac{W\, \pi\, (a + a_1)\, r_2}{2\, r_1}. \qquad \text{But, } a = a_1$$

$$\therefore \text{the moment} = \frac{W\, \pi\, a}{r_1}\, r_2$$

$$\text{Work per revolution, } Q = \frac{W\, \pi\, a}{r_1}\, r_2 \times 2\, \pi$$

Substituting the numerical values,

$$Q = \frac{10000 \times 3.14 \times .02 \times 1.625 \times 2 \times 3.14}{1.5 \times 12}$$

$$= 356.3 \text{ ft. lbs.}$$

PROBLEMS

244. 80 foot-pounds of work is required to drag a body weighing 10 lbs. up an inclined plane with the minimum force creating a vertical displacement of 4 feet. Find the inclination of the plane if the coefficient of friction be .2. *Ans.* 10° — 42′

245. The coefficient of friction between the nut and a square-threaded screw of 2.5 inches mean diameter and having two threads per inch of length is .02. What horizontal force should be effected at the circumference of the screw to lift a load of 5 tons? *Ans.* .412 ton.

246. To reduce the speed of a locomotive-engine brake-shoe is pressed against the surface of the wheels with a force of 1 ton, when it is running with a constant speed of 60 miles per hour. If the coefficient of friction between the wheels and the brakes be .25, find the horse-power absorbed by the brake. *Ans.* 89.6 H.P.

247. A train including locomotive weighs 150 tons, of which 35 tons act on the driving wheels. If the coefficient of adhesion be $\frac{1}{7}$, what is the greatest pull the engine can exert? With this pull, in what time will the train develop a speed of 30 miles per hour from rest against an average resistance due to air, friction, etc. of 20 lbs. per ton weight of the train?

Ans. 5 tons ; 55.99 secs.

248. One end of a 20-feet ladder rests against a rough vertical wall while the other rests on the ground making an angle of 60° with it. If the weight of the ladder be 50 lbs. and be assumed to act at a point of 9 feet from the lower end and if the ladder be just at the point of slipping, find the coefficient of friction between the ladder and the ground when that between the ladder and the wall is .25. *Ans.* .24.

249. A ladder is placed against a rough wall making an angle of 60° with the floor. A man climbs up the ladder. Find the position of the man up along the ladder beyond which an attempt to climb will disturb the condition of equilibrium. Neglect the weight of the ladder and take $\mu = .268$.

Ans. $.5\,l$.

250. In the previous problem if the weight of the ladder be taken into account, and if it be one-tenth of the weight of the man, will there be any change in the position? If any, find it. *Ans.* $.4984\,l$.

251. A ladder 14 feet in length has six equally spaced steps. If a man weighing 10 stones and 10 pounds starts to go up the ladder, determine the number of steps the man can pass without disturbing the state of equilibrium. If the man proceeds further to rise up, what portion of his weight will be required on the next step to attain the limiting condition of equilibrium? Angle of friction at both the ends is 15°. Neglect the weight of the ladder.

Ans. 2 steps ; pressure on the 3rd. step = 31.1 lbs.
pressure on the 2nd. step = 118.9 lbs.

252. Neglecting the weight of the ladder (Fig. 183) what will be the reactions at A and B if the angle of friction between the ladder and the wall

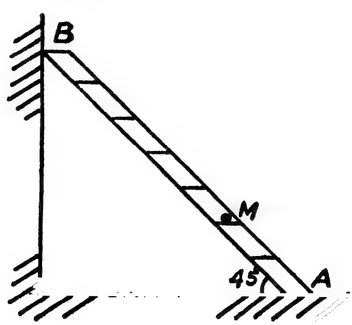


FIG. 183

and the floor be 22°, when a man weighing 10 stones is at M, which is at a distance of $\frac{2}{7}$ of the length of the ladder from the floor?

253. If the coefficient of friction between the beam and the supports (Fig. 133) be .3, find the horizontal reactions at A and B.

254. A uniform joist 20 feet long weighing 425 lbs. rests against a wall making an angle of 60° with the ground. A cord is attached with the ground end of the joist and drawn horizontally to reduce the inclination of the joist. If the coefficient of friction both at the ground and the wall be .5, find the least force applied through the cord to move the joist at that inclination.
Ans. 428 lbs.

255. In a jack-screw, the screw has 1.5 inches mean diameter with 2 square threads per inch. The length of the lever arm, measured from the axis of the screw, is 15 inches and the coefficient of friction is .05. If a force of 10 lbs. be applied at the end of the arm to rotate the screw, find the load lifted by the mechanism. *Ans.* 1281 lbs.

256. A square threaded jack-screw of 3 inches outside diameter has two threads per inch of length. It is rotated by a straight rod fixed to it whose effort end is at a distance of 4 feet from the axis of the screw. Determine the weight that can be lifted by an effort of 40 lbs., assuming $\mu = .15$ and neglecting the end friction of the screw. *Ans.* 595 lbs.

257. If in the problem 135 the surface of the table top is not perfectly smooth and the coefficient of friction, $\mu = .15$, $a = 5$ ft., $l = 20$ ft and $w = 1.5$ lbs. per foot length of the cord, find with what kinetic energy the cord will leave the table contact. What will be the velocity of the cord when it leaves the table? *Ans.* 225.94 ft. lbs.; 3.88 ft./sec.

258. A cow-boy is being dragged by a cow with a force of 50 lbs. Finding no other means to check the cow, he, while running by the side of a telegraph post, gives two and a half turns of the rope round the post and pulls his end of the rope with a force of 6.5 lbs. What is the coefficient of friction between the rope and the post? *Ans.* .13.

259. A block of ice weighing 200 lbs. is suspended from a fixed pulley with a flat band and kept in balance by hanging a weight from the other end. Compute the quantity of ice melted away to break the state of equilibrium. The band laps half round the pulley and $\mu = .25$. *Ans.* 108.8 lbs.

260. A lifting mechanism similar to wheel and axle, has a pulley of 30 inches diameter and a drum of 10 inches diameter fitted on a shaft 2 inches in diameter (Fig. 184). If a load, $W = 750$ lbs. be raised with uniform velocity by wrapping the cord round the drum surface and if the effort P on the pulley be in the directions as shown by the full and dotted lines

with arrow-heads, find P in the two cases. Neglect the consideration of the rope diameter and take the value of μ as .15.

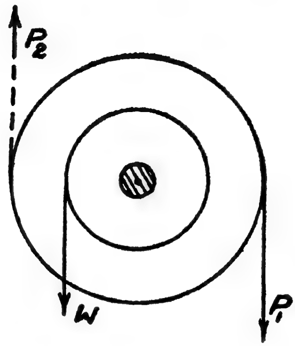


FIG. 184

Ans. 260 lbs.; 255 lbs.

261. In the previous problem if the load is lowered with uniform velocity, find the tension maintained in the rope at the effort end.

Ans. 240 lbs.; 245 lbs.

262. If in problem 260 the load is lowered with uniform retardation of 1 foot per second per second, find the pull of the rope at the effort end.

Ans. 247.8 lbs.; 252.7 lbs.

263. Figure 185 represents a brake arrangement. The strap round the pulley is $\frac{3}{16}$ inch thick. In the lever *ABC*, *A* is the effort end, *B* is the fulcrum and *C* is the load end. The mechanism is so arranged that the strap

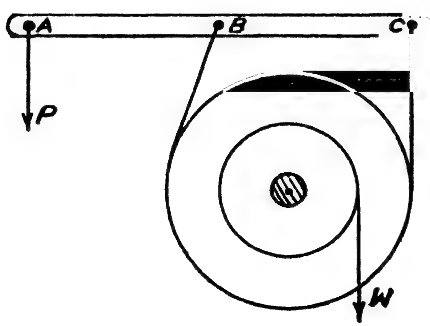


FIG. 185

goes vertically downwards from the point *C*. The pulley diameter is 20 inches and the drum diameter round which the lifting rope of $\frac{3}{4}$ inch in diameter

is coiled is 10 inches. Both the pulley and the drum are fixed on the same axle. If B is the middle of the lever which is 30 inches long and the strap length in contact with the pulley surface subtends an angle of 200° and if $\mu = .25$, find the weight of mass W that can be held up by the application of a force $P = 100$ lbs. at right angles to the lever arm.

Ans. 109.3 lbs.

264. If in the previous problem the fulcrum is shifted so that $AB : BC = 2$, and if the strap be in contact with a portion of the pulley surface which subtends an angle of 230° at the centre, what is the value of W?

Ans. 218.6 lbs.

265. If in the previous problem, where $AB : BC = 2$, the weight is applied on the other side of the drum, find its value. *Ans.* 523 lbs.

266. If in problem 264 the axle diameter be 2 inches and $\mu = .15$, find the value of W. *Ans.* 221.5.

267. A vertical shaft weighing 7,500 lbs. is fitted on a flat pivot on which the intensity of pressure (taken to be uniform) is 150 lbs per square inch. If the speed of the shaft is 200 r.p.m. and if the coefficient of friction be .008, find the horse power absorbed by the pivot. *Ans.* .506.

268. In the previous problem what will be the horse power absorbed from the consideration of uniform wear? *Ans.* .379.

269. The propeller shaft of a sea-going vessel contains 10 collars of 24 inches diameter. If the diameter of the shaft is 14 inches and its speed is 100 r.p.m., find the loss of power due to collar friction. Take, $\mu = .05$ and the intensity of thrust as uniform of value of 50 lbs. per sq. inch.

Ans. 115 H.P.

270. The length of a shaft is l and its diameter is d. If the density of the material be w per unit volume and if the shaft rests on a V-grooved guide, find the pull along the axis to move it horizontally. The coefficient of friction is μ and the angle of the groove is θ.

Ans. $$\frac{\mu \pi d^2 l w}{4 \sin \frac{\theta}{2}} = \frac{\mu W}{\sin \frac{\theta}{2}},$$

where W is the weight of the shaft.

271. The weight of an axle is W lbs. and it rests on two bearing surfaces equally spaced on either side of the vertical plane of symmetry of the axle as shown (Fig. 186). If the coefficient of friction be μ, find the tangential force applied at C, to give just a start to the rotational motion of the axle. For what value of α, the device is self-locking *i.e.*, P becomes infinity to give the start?

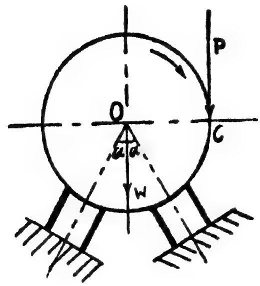


FIG. 186

Ans. $P = \dfrac{\mu W}{\cos\alpha\,(1+\mu^2) - \mu}$, $\alpha = \cos^{-1}\dfrac{\mu}{1+\mu^2}$

272. A fly-wheel weighing 1000 lbs. is supported on a shaft 5 inches in diameter and is placed at equal distances from the two bearings. The resultant of the weights of the composing parts of the fly-wheel acts at a point 1 inch away from the axis (which should be on the axis). If the fly-wheel rotates 300 times per minute and if the coefficient of friction between the journal and the bearing surfaces is .0322, determine the horse power necessary to overcome the resistance due to the friction in the bearings, which is a mere loss due to error of fixing the centre of the weight. *Ans.* 1.53 H.P.

273. A 6-in. belt transmits 8 H.P. The pulley on which it is fitted is 20 inches in diameter and runs at 200 r.p.m. If the maximum allowable tension is 80 lbs. per inch width of the belt and if it is assumed that the sum of the tensions at the tight and the slack sides is always constant, find the initial tension in the belt with which it is fitted. If $\mu = .25$, what is the measure of the angle of grip? *Ans.* 170.6°.

274. In a belt and pulley connection the angle of grip is 180° and $\mu = .3$. The pulley runs at 300 r.p.m. and has a diameter of 30 inches. Neglecting the centrifugal tension find the width of the single leather belt used to transmit 10 H.P. The maximum allowable tension in single leather belt is 40 lbs. per inch width. *Ans.* 3½ inches.

275. In the previous problem if the effect of the centrifugal tension is taken into account, what change in the width of the belt should be made? 12″ × 1″ single belt weighs .08 lb. *Ans.* 4½ inches.

276. If a hemp rope is fitted round a V-grooved fixed pulley and laps half the pulley surface, and if a weight of 200 lbs. is suspended, find what least effort at the other end of the rope will be able to give the weight an upward motion. $\mu = .3$. *Ans.* 2344 lbs.

CHAPTER IX

CENTRE OF GRAVITY

224. Centre of Mass. It is also called the *Centre of Inertia* or the *Mass Centre.* In case of a rigid body, where the constituent particles are in different planes, if each of the particles is acted upon by a force relative to the mass in the same direction then all the forces being parallel, the centre of this system of parallel forces which is quite independent of the direction of the forces is called the centre of the mass or the centre of inertia or the mass centre of the body. If M be the total mass of the body and m_1, m_2, m_3, etc. be the masses of the constituent particles respectively, then the forces proportional to the masses can be represented by m_1, m_2, m_3, etc. respectively and their resultant by M.

Thus, $M = m_1 + m_2 + m_3 + \cdots = \Sigma m$

If $\bar{x}$, $\bar{y}$ and $\bar{z}$ be the distances of the centre of mass of the body with respect to the three rectangular co-ordinate planes, X, Y and Z respectively, then, $\bar{x} = \dfrac{\Sigma mx}{\Sigma m}$, $\bar{y} = \dfrac{\Sigma my}{\Sigma m}$, $\bar{z} = \dfrac{\Sigma mz}{\Sigma m}$, where Σmx, Σmy and Σmz represent the sum of the plane moments of the forces about the three planes respectively.

225. Centre of Gravity. In particular cases where the volumes of the bodies are very small in comparision with the volume of the earth, the weights of the constituent particles, which are proportional to their masses and act towards the centre of the earth, may be taken for all practical purposes, to form a system of parallel forces. The centre of this system is called the *Centre of Gravity or Centroid* of the body. It is to be noted that this centre is also the mass centre of the body. It is also to be marked that if a force equal to the weight of the body be applied at that centre just in the opposite direction *i.e.*, in the vertical upward direction the body must remain at rest at any position in space whatsoever. The term *Centre of Gravity* is always denoted by the letters C.G., the initials.

When the volume of a body is big enough in relation to that of the earth, all the weights of the individual particles composing the

whole body cannot be considered as parallel. In that case the body cannot be said to have a centre of gravity, though it has always a centre of mass.

226. C.G. of a Body. Let M be the mass of the body and m_1, m_2, m_3, etc. be the masses of the different constituent particles respectively. Then,

$$M = m_1 + m_2 + m_3 + \cdots = \Sigma\, m$$

The position of C.G. with reference to the three mutually perpendicular planes will be such that,

$$x = \frac{\Sigma\, m\, x}{M}\,,\ \bar{y} = \frac{\Sigma\, m\, y}{M} \text{ and } \bar{z} = \frac{\Sigma\, m\, z}{M} \quad \cdots\cdots\cdots Eq.\ 139$$

Weight being proportional to mass, the forms may also be put in the following way. If W be the weight of the body and w_1, w_2, w_3, etc., be the weights of the constituent particles.

$$x = \frac{\Sigma\, w\, x}{W}\,,\ \bar{y} = \frac{\Sigma\, w\, y}{W} \text{ and } \bar{z} = \frac{\Sigma\, w\, z}{W}$$

227. C.G. of a System of Bodies. If the centres of gravity of different bodies in a system are known, then, the system can be treated just in the same way as was done in case of a single body. Each body in the system can be taken as a constituent particle in a body and hence, if the total weight of the system,

$$W = w_1 + w_2 + w_3 + \cdots = \Sigma\, w,$$

where w_1, w_2, w_3, etc. are the weights of the separate bodies respectively and $\Sigma\, w$ represents their sum, then,

$$x \quad \Sigma\, w = w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots = \Sigma\, wx$$

Therefore, $\bar{x} = \dfrac{\Sigma\, w x}{\Sigma\, w}$, where x represents the distance of the centre of the system from the Y-plane. Similarly, the distance from the X-plane, $\bar{y} = \dfrac{\Sigma\, w\, v}{\Sigma\, w}$ and the distance from the Z-plane, $\bar{z} = \dfrac{\Sigma w z}{\Sigma\, w}$

Here, the distances of the centres of gravity of the individual bodies from the three planes are represented by (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), etc. respectively.

228. C.G. of a Homogeneous Body. In a homogeneous body the weight per unit volume is constant. If w be the weight per unit volume (Fig. 221) and if dV be the volume of an elementary particle, then, the total weight of the body, $W = w \int dV$, and,

$$\bar{x} = \frac{w \int dV.x}{w \int dv} = \frac{\int dV.x}{V}, \quad \bar{y} = \frac{\int dV.y}{V} \text{ and } \bar{z} = \frac{\int dV.z}{V}$$

In terms of weight the equations can be put as,

$$x = \frac{\int dV.w.x}{\int dV.w} = \frac{\int dW.x}{W}, \quad y = \frac{\int dW.y}{W}, \quad \bar{z} = \frac{\int dW.z}{W}$$

As weight is proportional to the mass, the equations in terms of mass will be,

$$\bar{x} = \frac{\int dM.x}{M}, \quad \bar{y} = \frac{\int dM.y}{M} \text{ and } z = \frac{\int dM.z}{M}$$

In determining the moments care must always be taken in measuring the distances from the planes. The distances to the right of the X-plane, or above the Y-plane or in front of Z-plane are considered as positive, and otherwise as negative.

229. C.G. of a Volume or Mass from which a portion has been removed.

Following the arguments in article 227, if a body be divided into two parts, M_1 and M_2, *i.e.*, V_1 and V_2, then,

$$(V_1 + V_2)\ x = V_1\ x_1 + V_2\ x_2$$

where x is the distance of the C.G. of the whole system and x_1 and x_2 are the distances of the centres of gravity of V_1 and V_2 respectively.

If $V_1 + V_2$ be represented by V, then,

$$V_1\bar{x}_1 = Vx - V_2x_2$$

or, $$\bar{x}_1 = \frac{V\bar{x} - V_2\bar{x}_2}{V_1}, \text{ i.e., } x_1 = \frac{V\bar{x} - V_2\bar{x}_2}{V - V_2}$$

In terms of mass,

$$\bar{x}_1 = \frac{M\bar{x} - M_2\bar{x}_2}{M - M_2}, \text{ where } M \text{ is the total mass.}$$

230. C.G. of a Thin Rod of uniform section and material :

Case 1. STRAIGHT ROD. Take the two rectangular co-ordinate axes in such a way that the X-axis coincides with the centre line of the rod and Y-axis passes through one end as shown (Fig. 187). If w be the weight per unit length of the rod,

$$\bar{x} = \frac{\int_0^l w\,dx\,x}{\int_0^l w\,dx} = \frac{\int_0^l x\,dx}{\int_0^l dx}$$

$$= \frac{l}{2} \qquad \cdots\cdots\text{Eq. } 140$$

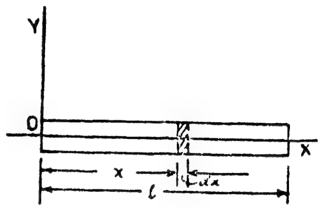


FIG. 187

where l is the length of the rod and dx is an elementary length of it at a distance x from the Y-axis.

Because the rod has been said to be thin and uniform, by reducing the cross-sectional area uniformly and gradually, the limiting position stands as a straight line, which is nothing but the centre line of the rod. Therefore, it is evident that the C.G. of the straight rod must be at the middle of the centre line of the rod.

Case 2. ROD, the centre line of which is a plane curve :—

Let AB (Fig. 188) be the centre line of such a rod, and w be the weight per unit length. Take any two rectangular axes OX and OY, in the same plane with the centre line of the rod. The weight of any elementary length dl is $w.dl$. If the co-ordinates of this elementary portion are x, y, then,

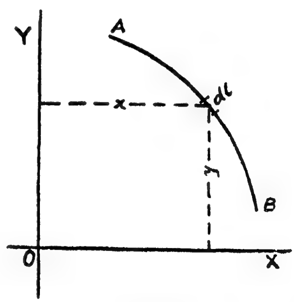


FIG. 188

$$\bar{x} = \frac{\int w\,x\,dl}{\int w\,dl} = \frac{\int x\,dl}{\int dl}$$

and $$\bar{y} = \frac{\int y\,dl}{\int dl}. \qquad \cdots\cdots\cdots\cdots Eq.\ 141$$

231. C.G. of a Line. Thus, by the term 'C.G. of a line' is meant the point which is nothing but the C.G. of a thin rod of uniform section and material whose centre line coincides with the given line.

232. C.G. of a System of Lines. The method is just similar to that of a system of bodies. If the lines are in the same plane and if the co-ordinates of their centres of gravity are known, then,

$$\bar{x} = \frac{\Sigma l x}{\Sigma l} \text{ and } \bar{y} = \frac{\Sigma l y}{\Sigma l} \quad \cdots\cdots\cdots\cdots Eq.\ 142$$

where $\bar{x}$ and $\bar{y}$ are the co-ordinates of the C.G. of the system of lines.

If the lines be in different planes, then the co-ordinates of the C.G. of the system will be such that,

$$\bar{x} = \frac{\Sigma l x}{\Sigma l},\ \bar{y} = \frac{\Sigma l y}{\Sigma l} \text{ and } \bar{z} = \frac{\Sigma l z}{\Sigma l} \quad \cdots\cdots\cdots\cdots Eq.\ 143$$

233. Deduction of Equations for C. G. of a Regular Plane Curve. C. G. of a circular arc. Let AB be the arc and OX and OY be the rectangular co-ordinate axes in the same plane with it and let the origin O be the centre of the arc whose radius is r (Fig. 189). Consider an elementary length dl at an angle θ with the axis OX and which subtends an angle $d\theta$ at O. Then,

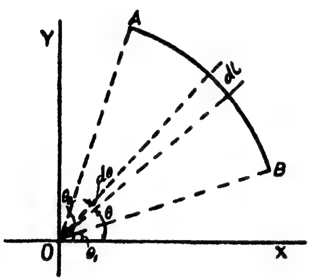


FIG. 189

$$dl = r.d\theta \text{ and, } x = r\cos\theta$$
$$y = r\sin\theta$$

where x and y are the distances of dl from the Y-axis and X-axis respectively.

Now, if $\bar{x}$ and $\bar{y}$ be the co-ordinates of the C. G. of the arc, then,

$$\bar{x} = \frac{\int x\, dl}{\int dl} \quad \text{and } \bar{y} = \frac{\int y\, dl}{\int dl}$$

If the lines OB and OA make angles θ_1 and θ_2 respectively with OX, then,

$$\bar{x} = \frac{\int_{\theta_1}^{\theta_2} r^2 \cos\theta\ d\theta}{\int_{\theta_1}^{\theta_2} r\ d\theta} = \frac{r(\sin\theta_2 - \sin\theta_1)}{\theta_2 - \theta_1}$$

$$\text{and} \quad y = \frac{\int_{\theta_1}^{\theta_2} r^2 \sin \theta \, d\theta}{\int_{\theta_1}^{\theta_2} r \, d\theta} = \frac{r (\cos \theta_1 - \cos \theta_2)}{\theta_2 - \theta_1}$$

............Eq. 144

(a) When $\theta_1 = 0$, $\bar{x} = \frac{r \sin \theta_2}{\theta_2}$, and $y = \frac{r (1 - \cos \theta_2)}{\theta_2}$

............Eq. 145

(b) When the arc is symmetrically divided into two parts by the X-axis, the subtended angle will also be divided into two equal parts. If the measure of each part be θ_2, then, below X-axis the angle is taken as negative and above the axis as positive. Thus,

$$x = \frac{r \left| \sin \theta \right|_{-\theta_2}^{+\theta_2}}{\left| \theta \right|_{-\theta_2}^{+\theta_2}} = \frac{r \sin \theta_2}{\theta_2}$$

$$\text{and} \quad y = \frac{-r \left| \cos \theta \right|_{-\theta_2}^{+\theta_2}}{\left| \theta \right|_{-\theta_2}^{+\theta_2}} = 0$$ ....Eq. 146

That is, the C.G. is on the X-axis at a distance of $\frac{r \sin \theta_2}{\theta_2}$, $\bar{z}$ in all these cases is equal to Zero.

(c) Semi-circular arc :—Placing the arc as in case (b)

$$\bar{x} = \frac{r \sin \theta}{\theta} = \frac{r \sin \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2r}{\pi}$$ ....Eq. 147

that is, the distance of the C.G. is $\frac{2r}{\pi}$ along the radius at right angles to the diameter of the semicircle.

Illus. Ex. 116. *Fig. 231 (VI) represents a system of three lines—two arcs and a straight line as shown. Find the C.G. of the system.*

From Eq 145 $\bar{y} = \dfrac{r\ (1 - \cos\theta)}{\theta}$

Therefore, for the whole system,

$$\bar{y} = \frac{\dfrac{6\,(1 - \cos 60)}{\pi} \times 6 \times \dfrac{\pi}{3} \times 2}{\dfrac{3}{\left(2 \times 6 \times \dfrac{\pi}{3}\right) + 6}}$$

$= 1\ 94$ inches.

$\bar{x} = 0$, the moment of the right hand portion is positive, whereas that of the left hand portion is negative But both of them are equal in magnitude

It is to be marked that the Y-axis divides the system symmetrically into two portions

234. C.G. of a Lamina with uniform thickness and material. It is evident from the foregoing discussions that the C G. must lie on a plane surface passing exactly through the middle of the thickness of the lamina. Let that plane be the Z-plane of the three rectangular co-ordinate planes, and refer the case to two rectangular co-ordinate axes, OX and OY, on that plane (Fig. 190). Let t be the thickness of the lamina, w be the weight per unit volume and dA be an elementary area of the surface. Then,

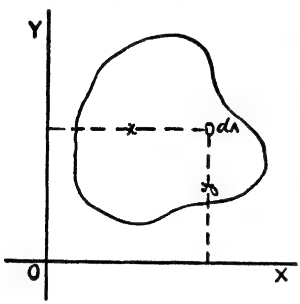


FIG. 190

$$\bar{x} = \frac{\int t.w.dA.x}{\int t.dA.w} = \frac{\int dA.x}{A}, \quad \bar{y} = \frac{\int dA.y}{A} \text{ and } \bar{z} = 0$$

.........*Eq.* 148

C.G. OF A PLANF SURFACE. Actually a mass can have a centre of gravity. But in computation often the relations obtained in *Eq.* 148,

$\bar{x} = \dfrac{\int dA.y}{A}$ and $\bar{y} = \dfrac{\int dA.x}{A}$, are found to occur which are just similar in forms to the equations used to find out the position of C.G. of a mass. The only difference in the forms is that in place of mass (M) area (A) has been found to occur. These

relations are said to denote the position of C.G. of an area, and we use the term "Centre of Gravity of an Area" without any hesitation.

It is to be noted that if the axes pass through the C.G. of the area or the mass, then, $\bar{x}$, $\bar{y}$ or $\bar{z}$, as the case may be, which are nothing but the distances of C.G. from the axes, being equal to zero, $\int dA.x = 0$, $\int dA.y = 0$ and $\int dA.z = 0$ and $\int dM.x = 0$, $\int dM.y = 0$ and $\int dM.z = 0$. That is, the moment of an area or a mass about an axis passing through the C.G. is equal to zero.

235. The moment of an area or a body may be positive as well as negative and that depends on the position of the rectangular co-ordinate axes of reference with respect to the given area or body. It may be that the axes occupy a position such that the whole of the area or body is distributed in four quadrants. Then, in the *first* and *second* quadrants the moments about the X-axis are positive, while in the *third* and *fourth* quadrants they are negative, according to the convention of measuring the distance from the axis. Similarly, the moments about the Y-axis in the *first* and *fourth* quadrants are positive and in the *second* and *third* quadrants negative. Whether the grand total is a positive quantity or not depends on the numerical values of the four moments in the four quadrants mentioned above. The position of the C.G. of the area, ($\bar{x}$, $\bar{y}$) *i.e.*, the distances from the axes and the position in space, whether to the right or left of the Y-axis, or up or down the X-axis, can be determined from the result of $\dfrac{\text{Total Moment}}{\text{Total Area}}$.

236. C.G. of a System of Plane Surfaces. If the centres of gravity of individual areas are known,

$$\bar{x} = \frac{A_1 x_1 + A_2 x_2 + A_3 x_3 + \dots\dots}{A_1 + A_2 + A_3 + \dots\dots} = \frac{\Sigma Ax}{\Sigma A},$$

where A_1x_1, A_2x_2, etc. are plane moments of the areas about the X-plane.

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A}$$

237. C.G. of an area with a part removed. Let the total area be A_1 and the area of the part removed be A_2 (Fig. 191). Therefore,

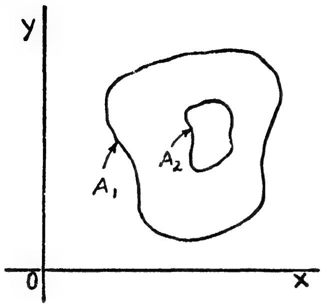


FIG 191

the actual area $A = (A_1 - A_2)$. If the co-ordinates of the centres of gravity of A_1, A_2 and A be (x_1, y_1), (x_2, y_2) and $(\bar{x}, \bar{y})$ respectively, then,

$$A_1 x_1 = \int x.dA_1 \text{ and } A_1 y_1 = \int y.dA_1$$
$$A_2 x_2 = \int x.dA_2 \text{ and } A_2 y_2 = \int y.dA_2$$

and

$$A.\bar{x} = \int x.dA \quad \text{and} \quad A.\bar{y} = \int y.dA$$

But, $\int x.dA + \int x.dA_2 = \int x.dA_1$.

Therefore, $A.\bar{x} + A_2 x_2 = A_1 x_1$

Hence, $\bar{x} = \dfrac{A_1 x_1 - A_2 x_2}{A}$, similarly, $\bar{y} = \dfrac{A_1 y_1 - A_2 y_2}{A}$

Eq. 149

238. Deduction of Formulae for determining the position of C. G. of Regular Plane Surfaces.

Case I. Rectangle and Parallelogram.

It is to be noted that in case of a plane surface the reference of two rectangular co-ordinate axes are quite sufficient to give a definite position in space of the C G. of the surface.

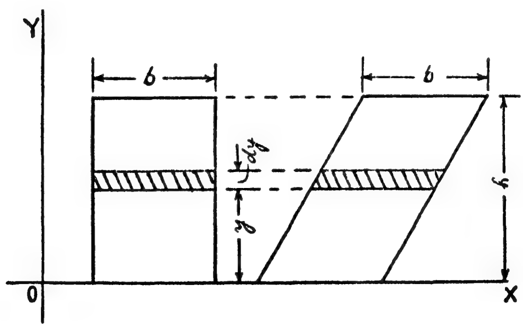


FIG. 192

Let the X-axis coincide with the smaller side of the area in both the surfaces. Also let h be the altitude and b be the length of the

smaller side (Fig. 192). Take a strip as an elementary area at a distance y from the X-axis and parallel to it. Let its width be dy, then, the distance of the C.G. from the X-axis,

$$\bar{y} = \frac{\int y.dA}{\int dA}, \text{ here,}$$

$$= \frac{\int_0^h y.\,b.\,dy}{A} = \frac{b\int_0^h y.\,dy}{b.h} = \frac{h}{2}$$

Since the C.G. of each strip is at the centre of that strip, the C.G. of the whole area must lie on the straight line passing through the middle points of the strips, *i.e.*, on the straight line joining the mid-points of the two smaller sides of the figures. Therefore, the C.G. is at the half distance of this line.

Similarly, it can be proved that the C.G. is also at the half distance of the line joining the mid-points of the two bigger sides of the areas.

Hence, the C.G. is at the crossing of these two lines, *i.e.*, at the intersecting point of the two diagonals.

Case II. Triangle.

(*a*) The distance of the C.G. from the base.

Let b be the base and h be the altitude of the triangle ABC (Fig. 193). Draw two rectangular axes, OX and OY, in such a

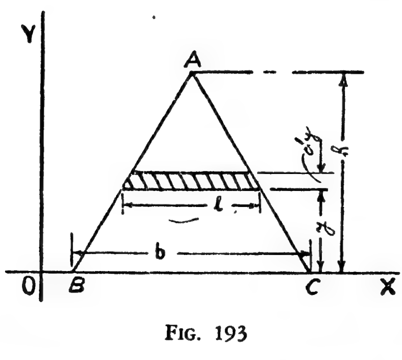


FIG. 193

way that OX coincides with the base. Take an elementary strip of the area, dA, parallel to the base, whose length is l and width is dy at a distance y from the base. From the similarity of traingles,

$$\frac{l}{b} = \frac{h - y}{h}, \text{ or, } l = \frac{b\,(h - y)}{h}$$

But $d\,A = l \times d\,y$, and therefore, equal to $\frac{b\,(h - y)}{h} \cdot dy$

$$\therefore \quad \bar{y} = \frac{\int y\,dA}{A} = \frac{\int_0^h \frac{b\,(h - y)}{h}\,y\,dy}{\frac{b\,h}{2}}$$

$$= \frac{\int_0^h \left(b - \frac{b}{h}\,y\right) y\,dy}{\frac{b\,h}{2}}$$

$$= \frac{\int_0^h b\,y\,dy - \int_0^h \frac{b}{h}\,y^2\,dy}{\frac{b\,h}{2}} = \frac{\frac{bh^2}{2} - \frac{bh^3}{3h}}{\frac{bh}{2}} = h - \tfrac{2}{3}\,h = \tfrac{1}{3}\,h.$$

Since the C.G. of each elementary strip is in its middle point, the C.G. of the area must be on the median. Hence, the ditsance of the C.G. from any side of a triangle is $\frac{1}{3}$ of the corresponding median along it from the side, *i.e.*, the C.G. of a triangle is at the intersection of the medians of the triangle.

(*b*) The distance of the C.G. from an axis passing through the vertex and parallel to the base.

It is evident from the previous proof that the distance is vertically $\frac{2}{3}\,h$ from the vertex, *i.e.*, from the axis referred to.

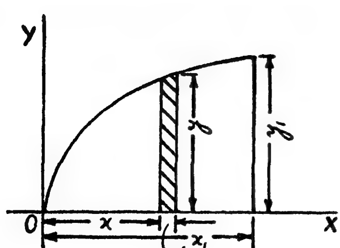


FIG. 194

Case III. Half of a Parabolic Area.

From two rectangular co-ordinate axes one of which, OX, coincides with the line that symmetrically bisects the parabolic area, and the other, OY, passes through the vertex. The equation of a parabolic curve is $y^2 = m\,x$. Hence, $y = m^{\frac{1}{2}}\,x^{\frac{1}{2}}$. Consider an elementary

strip of the area (Fig. 194) which is parallel to the Y-axis at a distance x from it and having a width of dx. If the length of the strip be y and its area be dA, then, $dA = y.\,dx$.

$$\text{Thus, } \bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int y\,dx.\,x}{\int y\,dx} = \frac{\int_0^{x_1} m^{\frac{1}{2}} x^{\frac{3}{2}}\,dx}{\int_0^{x_1} m^{\frac{1}{2}} x^{\frac{1}{2}}\,dx} = \frac{\int_0^{x_1} x^{\frac{3}{2}}\,dx}{\int_0^{x_1} x^{\frac{1}{2}}\,dx}$$

$$= \frac{3}{5}\,x_1$$

$$\text{And, } \bar{y} = \frac{\int_0^{x_1} \frac{y}{2}\,.\,y\,dx}{\int_0^{x_1} y\,dx} = \frac{\frac{m}{2}\int_0^{x_1} x\,dx}{m^{\frac{1}{2}}\int_0^{x_1} x^{\frac{1}{2}}\,dx} = \frac{3}{8}\,m^{\frac{1}{2}} x_1^{\frac{1}{2}} = \frac{3}{8}\,y_1$$

Or,

If the strip be taken parallel to X-axis,

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int_0^{y_1} (x_1 - x)\,y\,dy}{\int_0^{y_1} (x_1 - x)\,dy} = \frac{\int_0^{y_1} \left(\frac{y y_1^2}{m} - \frac{y^3}{m}\right) dy}{\int_0^{y_1} \left(\frac{y_1^2}{m} - \frac{y^2}{m}\right) dy}$$

$$= \frac{\frac{1}{m}\int_0^{y_1} (y y_1^2 - y^3)\,dy}{\frac{1}{m}\int_0^{y_1} (y_1^2 - y^2)\,dy} = \frac{\int_0^{y_1} y y_1^2\,dy - \int_0^{y_1} y^3\,dy}{\int_0^{y_1} y_1^2\,dy - \int_0^{y_1} y^2\,dy}$$

$$= \frac{\frac{y_1^4}{2} - \frac{y_1^4}{4}}{y_1^3 - \frac{y_1^3}{3}} = \frac{3}{8}\,y_1$$

(a) In case of a *parabolic area* if the two axes are taken in such a way that the X-axis bisects the area symmetrically and if the strips be taken parallel to the Y-axis, it is evident that the C.G. of each strip being at the middle point of it, the C.G. of the total area must be on the X-axis and it is, of course, at a distance of $\frac{3}{5}\,x_1$ from the Y-axis.

Case IV. (*a*) *One-fourth of an Elliptical Area.*

Draw the two axes, OX and OY, coinciding with the major and minor axes of the area respectively (Fig. 195).

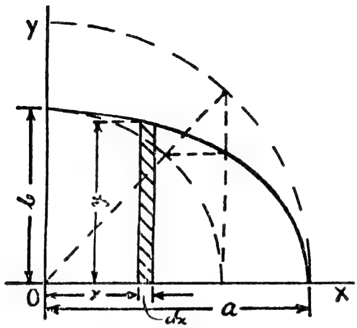


FIG. 195

Equation of an ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Therefore, $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, or, $y = \frac{b}{a}\sqrt{(a^2 - x^2)}$

Take a strip of area parallel to the Y-axis at a distance of x from it.

Let the width of the strip be dx, then, if dA be the area of the strip and y be its length, $dA = y.\,d\,x = \frac{b}{a}\sqrt{(a^2 - x^2)}.\,d\,x.$

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}.\,x\,dx}{\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx} = \frac{\int_0^a \sqrt{a^2 - x^2}\,x\,dx}{\int_0^a \sqrt{a^2 - x^2}\,dx}$$

$$= \frac{\left| -\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}} \right|_0^a}{\left| \frac{1}{2}x(a^2 - x^2)^{\frac{1}{2}} + \frac{1}{2}a^2 \sin^{-1}\frac{x}{a} \right|_0^a} = \frac{4a}{3\pi}$$

Similarly, it can be proved that,

$$\bar{y} = \frac{4b}{3\pi}.$$

Alternative Method

The problem can be solved with the help of parametric equations.

Remembering the method of developing an elliptical curve (Fig. 195) and denoting the two rectangular co-ordinate axes, X and Y, by the major and minor axes respectively, the co-ordinates of any point, p, on the curve are such that,

$$x = a \cos \theta \quad \text{and} \quad y = b \sin \theta,$$

where a and b represent half the major and minor axes respectively.

Now, the area of a strip of width dx at a distance x from the centre $= b \sin \theta . dx$.

Therefore, $\bar{x} = \dfrac{\int b \sin \theta \, dx . x}{\int b \sin \theta \, dx}$, but $x = a \cos \theta$

$\therefore \quad dx = - a \sin \theta \, d\theta$

Hence,

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} b \sin \theta \quad a \sin \theta \, d\theta . \, a \cos \theta}{\int_0^{\frac{\pi}{2}} b \sin \theta - a \sin \theta . \, d\theta}$$

$$= \frac{- a^2 b \int_0^{\frac{\pi}{2}} \sin^2 \theta . \cos \theta \, d\theta}{- ab \int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta}$$

Let $\sin \theta = v$
then, $\cos \theta \, d\theta = dv$
$\therefore \int \sin^2 \theta \cos \theta \, d\theta$
$= \dfrac{\sin^3 \theta}{3}$
and,
$\cos 2\theta =$
$\cos^2 \theta - \sin^2 \theta$
$= 1 - 2 \sin^2 \theta$
$\therefore \sin^2 \theta =$
$\frac{1}{2}(1 - \cos 2\theta)$

$$= \frac{a \left| \dfrac{\sin^3 \theta}{3} \right|_0^{\frac{\pi}{2}}}{\frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2\theta) \, d\theta}$$

$$= \frac{a}{3} \div \frac{\pi}{4}$$

$$= \frac{4\,a}{3\,\pi}$$

(*b*) *Half of an Elliptical Area.*

The axes are chosen as before and, say, a portion on either side of the *Y*-axis is considered. Because the centre of gravity of a strip is at the middle of its length, the locus of these centres is the *X*-axis.

Therefore, $\bar{y} = 0$, $\bar{x}$ of course will have a value equal to $\frac{4\,a}{3\,\pi}$.

(*c*) *Elliptical Area.*

Arguing in the same way with the strips parallel to the two axes respectively, the C.G. of the area must lie on both the minor and major axes of the figure. Therefore, C.G. is the point of intersection of those axes, *i.e.*, the point representing the origin. The centre of an ellipse is its centre of gravity.

Case V. Circular Sector.

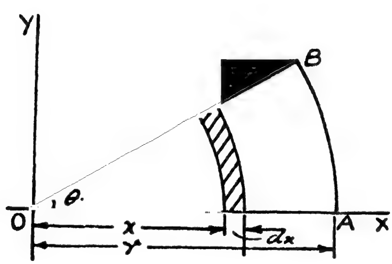


FIG. 196

Let the two axes be placed in such a way that X-axis coincides with a straight side of the sector as shown in the figure (Fig. 196), and let the origin be the centre of the circular arc. Also let the arc subtend an angle θ at the origin. Take a strip parallel to the arc at a distance x from the origin and let its width be dx. Then, dA, the area of the strip $= x.\,\theta.\,dx$. The C.G. of this strip from Y and X axes will be, $\frac{x \sin \theta}{\theta}$ and $\frac{x\,(1 - \cos \theta)}{\theta}$ respectively (*Eq* 145).

Therefore, if $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the whole area,

$$\text{then, } \bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^r x \frac{\sin\theta}{\theta} x\,\theta\,dx}{\int_0^r x\,\theta\,dx} = \frac{\sin\theta \int_0^r x^2\,dx}{\theta \int_0^r x\,dx}$$

$$= \frac{r^3}{3} \sin\theta \div \frac{r^2\theta}{2} = \frac{2}{3} r \frac{\sin\theta}{\theta}$$

$$\text{Similarly, } \bar{y} = \frac{2}{3} r \frac{1 - \cos\theta}{\theta}$$

(*a*) If a *circular sector* be bisected symmetrically by the X-axis, then, C.G. must lie on it at a distance of $\frac{2}{3} r \frac{\sin\theta_1}{\theta_1}$ from the origin, where θ is half the subtended angle. $\bar{y} = 0$.

(*b*) *Semi-circular Area.*

Let Y-axis coincide with the diameter, then,

$$\bar{x} = \tfrac{2}{3} r \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{4r}{3\pi} \text{ and } \bar{y} = 0.$$

Case VI. C.G. of a Circular Segment. (Fig. 197)

Let MPN be the segment cut off by the chord MN, subtending an angle 2θ at the centre. Select the axes in such a way that OX divides the segment into two exactly similar equal parts, and Y-axis passes through the centre. Then the C.G. must lie on the X-axis.

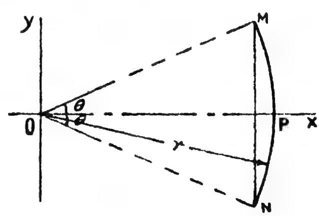


FIG. 197

The segment MPN = The sector $OMPN$ — the triangle OMN. Let the areas of the sector, the triangle and the segment be A_1, A_2 and A respectively. If $\bar{x}$, $\bar{y}$ be the co-ordinates of the C.G. of the segment and (x_1, y_1), (x_2, y_2) be those of the C.G.'s of A_1 and A_2 respectively,

then, $\bar{x} = \dfrac{A_1 x_1 - A_2 x_2}{A_1 - A_2}$

If r be the radius of the circular area,

$$A_1 = \pi r^2 . \frac{2\theta}{2\pi} = r^2 \theta$$

$$A_2 = \tfrac{1}{2} \times 2r \sin\theta \times r\cos\theta = r^2 \sin\theta \cos\theta$$

$$A_1 - A_2 = r^2\theta - r^2 \sin\theta \cos\theta = r^2(\theta - \sin\theta\cos\theta)$$

$$\bar{x} = \frac{\frac{2}{3} r \frac{\sin\theta}{\theta} \cdot r^2\theta - \frac{2}{3} r\cos\theta \times r^2 \sin\theta\cos\theta)}{r^2(\theta - \sin\theta\cos\theta)}$$

$$= \frac{2}{3} \frac{r^3 \sin\theta - r^3 \sin\theta\cos^2\theta}{r^2(\theta - \sin\theta\cos\theta)}$$

$$= \frac{2}{3} \frac{r^3 \sin\theta(1 - \cos^2\theta)}{r^2(\theta - \sin\theta\cos\theta)}$$

$$= \frac{2}{3} r \frac{\sin^3\theta}{\theta - \sin\theta\cos\theta}$$

239. C.G. of Composite Areas.

If a composite area can be divided into a number of finite parts, *i.e.*, simpler areas such as triangles, rectangles, etc., then, the centres of gravity of the individual areas of regular shape being known, the C.G. of the total area is determined by dividing the sum of the products of the individual areas and the distances of their respective centres of gravity from the axis of reference by the sum of the individual areas, *i.e.*, the total area.

Illus. Ex. 117. *Find the C.G. of the trapezoidal area (Fig. 198) with respect*

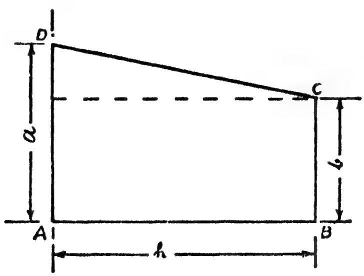


FIG. 198

to the sides AB and AD, coinciding with X and Y axes respectively, $a = 3''$, $b = 2''$ and $h = 6''$

(*I*) The area can be divided into a rectangle and a triangle by the dotted line as shown

$$\text{Then } \bar{x} = \frac{b\,h.\dfrac{h}{2} + \frac{1}{2}(a-b)h.\frac{1}{3}h}{b\,h + \frac{1}{2}(a-b)\,h}$$

$$= \frac{1}{3}h\left(\frac{a+2b}{a+b}\right)$$

Alternative method

At any distance x from AD the altitude of the area, y, is equal to

$$(a-b)\frac{h-x}{h} + b$$

If a differential strip of width dx be taken at a distance x from the side AD, its area will be equal to $\left\{b + (a-b)\dfrac{h-x}{h}\right\}dx$

$$\text{Hence, } \bar{x} = \frac{\displaystyle\int_0^h \left\{b + (a-b)\frac{h-x}{h}\right\} x\,dx}{A}$$

$$= \frac{1}{3}h\left(\frac{a+2b}{a+b}\right)$$

Similarly,

$$\bar{y} = \frac{1}{3}\,\frac{a(5b+a) - 3b^2}{a+b}$$

Substituting the numerical values, $\bar{x} = 2{\cdot}8$ inches, $\bar{y} = 1{\cdot}8$ inches

Illus Ex 118 *Locate the C G of the area given in Fig 199*

(1) Composite Area Method

The total area is divided into two rectangles—6 ins by $\frac{3}{4}$ in and $2\frac{1}{4}$ ins by $\frac{3}{4}$ in

$$\text{Total area} = \frac{9}{2} + \frac{27}{16} = \frac{99}{16} \text{ sq inches}$$

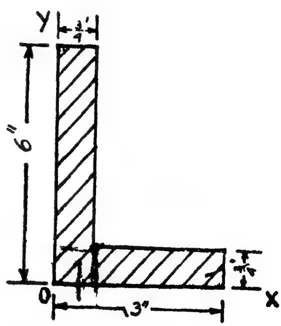


FIG 199

$$\therefore \quad \bar{x} = \left(\frac{9}{2} \times \frac{3}{8} + \frac{27}{16} \times \frac{15}{8} \right) \div \frac{99}{16} = \frac{621}{792}$$

$$= .784 \text{ inch.}$$

$$y = \left(\frac{9}{2} \times 3 + \frac{27}{16} \times \frac{3}{8} \right) \div \frac{99}{16}$$

$$= 2.284 \text{ inches.}$$

(2) Integration Method

$$x = \frac{\int_0^{\frac{3}{4}} 6x\,dx + \int_{\frac{3}{4}}^{3} \frac{3}{4}x\,dx}{\int_0^{\frac{3}{4}} 6\,dx + \int_{\frac{3}{4}}^{3} \frac{3}{4}\,dx}$$

$$= .784 \text{ inch.}$$

$$y = \frac{\int_0^{6} \frac{3}{4}\,y\,dy + \int_0^{\frac{3}{4}} 2\frac{1}{4}\,y\,dy}{\frac{99}{16}}$$

$$= 2.284 \text{ inches.}$$

(3) Part Removed Method

From the rectangular area, 6 ins. by 3 ins. a rectangular area 5¼ ins. by 2¼ ins. has been removed as is clear from the diagram.

$$\text{Hence, } x = \frac{6 \times 3 \times 1.5 - \frac{21}{4} \times \frac{9}{4}\left(\frac{3}{4} + \frac{9}{8}\right)}{18 - \frac{189}{16}}$$

$$= .784 \text{ inch.}$$

$$y = \frac{6 \times 3 \times 3 - \frac{21}{4} \times \frac{9}{4}\left(\frac{3}{4} + \frac{21}{8}\right)}{\frac{99}{16}}$$

$$= 2.284 \text{ inches.}$$

Illus. Ex. 119. *If in the area of the previous problem 3 inches side is changed to a length of 4 inches, determine what change in length of the other side be made so that the value of* $\bar{x}$ *remains unaltered.*

$$\bar{x} = \frac{621}{792} = \frac{y \cdot \frac{3}{4} \cdot \frac{3}{8} + \frac{13}{4} \cdot \frac{3}{4} \cdot \frac{19}{8}}{\frac{3y}{4} + \frac{39}{16}}$$, where y is the changed length of the side.

$$= \frac{36\,y + 741}{8\,(12\,y + 39)}$$

That is, $\dfrac{36\,y + 741}{12\,y + 39} = \dfrac{621}{99}$

From which, $y = 13.86$ inches.

Hence, the length should be increased by 7.86 inches.

240. Axis of Symmetry and C.G. It is clear from the foregoing discussions that if an axis divides an area into two symmetrical equal parts, the C.G. lies on that axis. Such an axis is called an *Axis of Symmetry.* If an area has two axes of symmetry, the C.G. lying on both the axes, must be at the point of intersection of the axes (Fig. 200).

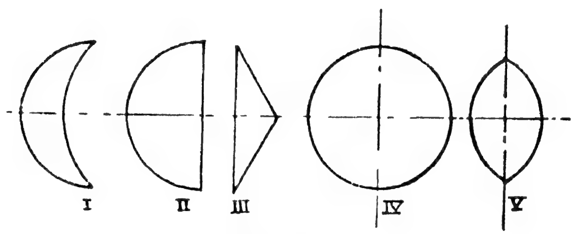


FIG. 200

241. Plane of Symmetry and C.G. If a plane divides a volume or mass into two identical equal parts, the plane is called the *Plane of Symmetry.* It is evident that all the plane sections of the volume or mass at right angles to the plane of symmetry are divided into two identical equal parts by the intersecting lines between the plane sections and the plane of symmetry, *i.e.*, the intersecting lines become the axes of symmetry for the sections. Thus, the centres of gravity of the plane sections lying on these intersecting lines, the C.G. of the volume or the mass must lie on the plane of symmetry formed by the intersecting lines. If there be two planes of symmetry for a volume or mass, it is clear the C.G., remaining on both the planes, must be on the line of intersection of the two planes. If there be more than

two planes of symmetry, the C.G. lying on all the planes, must be at the common point of meeting the planes.

242. Deduction of Formulae for determining C.G. of Homogeneous Bodies with regular shapes.

Case I. Right Circular Cone.

Let Fig. 193 represent the cone.

To find C.G. from the base. Let the X-axis coincide with the trace of the base area on a vertical plane. Take a strip parallel to the base at a distance y from it and of thickness dy. If A be the area of the base and A_1 be the surface area of the strip, then, the volume of the strip, $dV = A_1\,dy$ and, $\dfrac{A_1}{A} = \dfrac{(h-y)^2}{h^2}$ or, $A_1 = \dfrac{A(h-y)^2}{h^2}$

$$\therefore\ \bar{y} = \frac{\int y\,dV}{\int dV} = \frac{\int A_1 y\,dy}{\int A_1\,dy}$$

$$= \frac{\displaystyle\int_0^h \frac{A(h-y)^2}{h^2}\,y\,dy}{\displaystyle\int_0^h \frac{A(h-y)^2}{h^2}\,dy}$$

$$= \frac{\dfrac{A}{h^2}\displaystyle\int_0^h (h-y)^2\,y\,dy}{\dfrac{A}{h^2}\displaystyle\int_0^h (h-y)^2\,dy}$$

$$= \frac{\displaystyle\int_0^h (h^2y - 2hy^2 + y^3)\,dy}{\displaystyle\int_0^h (h^2 - 2hy + y^2)\,dy}$$

$$= \frac{\dfrac{h^4}{2} - \frac{2}{3}h^4 + \dfrac{h^4}{4}}{h^3 - h^3 + \dfrac{h^3}{3}} = \frac{\dfrac{6-8+3}{12}h^4}{\dfrac{h^3}{3}} = \frac{1}{4}h$$

From the vertex, therefore, the distance of C.G. $= \frac{3}{4}\,h$.

Volume of a cone $= \dfrac{A}{h^2}\displaystyle\int_0^h (h - y)^2\,dy = \tfrac{1}{3}\,A\,h = \tfrac{1}{3}\,\pi\,r^2\,h,$

where r is the radius of the base area.

Volume of a cone $= \frac{1}{3}$ base area $\times\, h$ ....*Eq.* 150

Case II. Cylinder with Plane Surfaces Parallel.

Let Fig. 192 represent the cylinders.

To find C.G. with reference to one of the plane surfaces. Let the X-axis coincide with the trace of the plane surface as before, and let h be the height of the cylinder. Consider a slice of the cylinder parallel to the base at a distance of y from it and of thickness dy. Then, the volume of the slice, $dV = A\,dy$, where A is the area of the plane face.

$$\bar{y} = \frac{\int dV\,y}{\int dV} = \frac{\int_0^h y\,dy}{\int_0^h dy} = \frac{h}{2} \qquad \text{............} Eq.\ 151$$

Volume of a cylinder $= \displaystyle\int_0^h A\,dy = A.\,h = \pi\,r^2\,h$, where r is the radius of the base area. ....*Eq.* 152

Case III. Segment of a sphere.

Assume the axes in such a way that each of the Y and Z planes divide the segment into two identical equal parts (Fig. 201), and the two axes pass through the centre of the sphere. Let ABC be the segment and let the trace on the Z-plane of the plane face of the segment, AC, subtend an angle 2θ at the origin. Let r be the radius of the sphere.

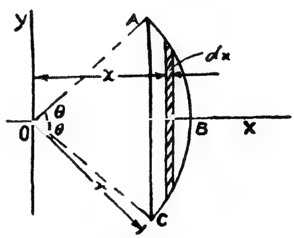


FIG. 201

If dV be the volume of a slice parallel to the X-plane at a distance x from the Y-axis and if dx be the width, then, $dV = \pi\,(r^2 - x^2)\,dx$.

Let a be the distance of the plane face of the segment from the Y-plane.

$$\text{Then, } \bar{x} = \frac{\int dV.x}{\int dV} = \frac{\int_a^r \pi(r^2 - x^2)\,x\,dx}{\int_a^r \pi(r^2 - x^2)\,dx} = \frac{3}{4}\cdot\frac{(r+a)^2}{(2r+a)^2}$$

$$= \frac{3}{4}\,\frac{r^2(1+\cos\theta)^2}{r\,(2+\cos\theta)} = \frac{3}{4}\,r\,\frac{(1+\cos\theta)^2}{2+\cos\theta}$$

$$\bar{y} = 0, \; \bar{z} = 0 \qquad \text{............Eq. 153}$$

Y and Z planes are the planes of symmetry.

$$\text{Volume of a segment} = \int_a^r \pi(r^2 - x^2)\,dx = \frac{\pi}{3}(r-a)^2(2r+a)$$

$$= \frac{\pi}{3}\,r^3\,(1-\cos\theta)^2\,(2+\cos\theta)$$

$$\text{............Eq. 154}$$

Case III (a). C.G. of a hemisphere.

Assuming the axes in such a way that Y-plane divides the solid into two identical equal halves and proceeding in the same way as was done in the previous case,

$$\bar{x} = \frac{3}{4}\,r\,\frac{(1+\cos\theta)^2}{2+\cos\theta}, \quad \theta \text{ being } \frac{\pi}{2} \text{ in this case,}$$

$$\bar{x} = \frac{3}{8}\,r$$

$$\bar{y} = 0$$

$$\bar{z} = 0 \qquad \text{............Eq. 155}$$

Y and Z planes are the planes of symmetry.

$$\text{The volume of the hemisphere} = \frac{\pi}{3}\,r^3\left(1-\cos\frac{\pi}{2}\right)^2\left(2+\cos\frac{\pi}{2}\right)$$

$$= \frac{2}{3}\,\pi\,r^3 \qquad \text{............Eq. 156}$$

Case IV. C.G. of a zone of a spherical shell.

By spherical shell is meant a hollow spherical body, the thickness of which is very small.

Y and Z planes are taken as the planes of symmetry in this case.

Let $ABCD$ be a zone parallel to the X-plane and let the angles BOX and AOX be θ_1 and θ_2 respectively (Fig. 202). Take an elementary angle $d\theta$ at the centre at an angular distance of θ from the Y-plane. Then, the width of the surface that subtends the angle $d\theta$ is $r.d\theta$, where r is the radius of the sphere. Now, the element of area, dA, of the zone surface subtending an angle $d\theta$ is equal to $2\,\pi .\, r.\sin\theta .\, r.\, d\theta = 2\,\pi\, r^2 \sin\theta .\, d\theta$. Therefore, the total area of the zone surface, $A = \int_{\theta_1}^{\theta_2} 2\,\pi\, r^2 \sin\theta\; d\theta$. If the thickness of the shell be t, then the volume of the elementary slice $= 2\,\pi\, r^2 \sin\theta\; d\theta\; t$.

Now, if $\bar{x}$ be the distance of the C.G. from the X-plane,

$$\bar{x} = \frac{\int_{\theta_1}^{\theta_2} 2\,\pi\, r^2 \sin\theta\; d\theta\; t\; x}{\int_{\theta_1}^{\theta_2} 2\,\pi\, r^2 \sin\theta\; d\theta\; t},$$

where x is the distance of the elementary slice from X-plane.

$$= \frac{\int_{\theta_1}^{\theta_2} \sin\theta\; d\theta\; r\cos\theta}{\int_{\theta_1}^{\theta_2} \sin\theta\; d\theta} = \tfrac{1}{2}\, r\,(\cos\theta_1 + \cos\theta_2)$$

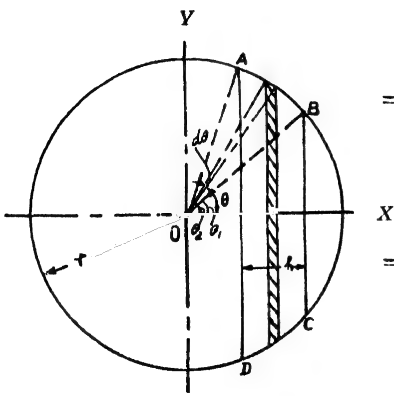


FIG. 202

$$= \frac{x_1 + x_2}{2},$$ where x_1 and x_2 are the distances of the faces AD and BC of the zone from X-plane.

$$= \frac{x_1 + x_1 + h}{2} = x_1 + \frac{h}{2},$$

where h is the width of the zone.

That is, the C.G. is at the middle of the width of the zone. $\bar{y}$ and $\bar{z}$ being equal to zero, the centre of gravity is at a distance of $x_1 + \frac{h}{2}$ from the centre along the X-axis.

Note : Zones of equal width have equal surface areas.

The area of the surface of a zone,

$$A = \int_{\theta_1}^{\theta_2} 2\pi r^2 \sin\theta \, d\theta = 2\pi r^2 \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta$$

$$= 2\pi r^2 (\cos\theta_2 - \cos\theta_1) = 2\pi r h \qquad \cdots\cdots \textit{Eq. } 157$$

Therefore, h remaining constant the area will also remain constant.

Case V. C.G. of a sector of a sphere. (Fig. 203)

Assuming the rectangular co-ordinate planes in the same way as was done in the last case, the sector is divided into two parts — a cone *AOC* and a segment *ABC*. Then, the distance of C.G. from the X-plane will be such that,

$$\bar{x} = \frac{Vol.\ AOC \times x_1 + Vol.\ ABC \times x_2}{Vol.\ (AOC + ABC)}$$

where x_1 and x_2 are the distances of the centres of gravity of the volumes respectively from the X-plane.

$$\text{Thus, } \bar{x} = \frac{\frac{\pi}{3} r^2 \sin^2\theta \times r\cos\theta \times \frac{3}{4} r\cos\theta + \frac{\pi}{3} r^3 (1-\cos\theta)^2 (2+\cos\theta) \times \frac{3}{4} r \frac{(1+\cos\theta)^2}{2+\cos\theta}}{\frac{\pi}{3} r^2 \sin^2\theta \times r\cos\theta + \frac{\pi}{3} r^3 (1-\cos\theta)^2 (2+\cos\theta)}$$

$$= \frac{\frac{1}{4} r^4 \sin^2\theta \cos^2\theta + \frac{1}{4} r^4 (1-\cos^2\theta)^2}{\frac{1}{3} r^3 \sin^2\theta \cos\theta + \frac{1}{3} r^3 (1-\cos\theta)^2 (2+\cos\theta)}$$

$$= \frac{3}{4} r \frac{\sin^2\theta\cos^2\theta + (1 - \cos^2\theta)^2}{\sin^2\theta\cos\theta + (1 - \cos\theta)^2\,(2 + \cos\theta)}$$

$$= \frac{3}{4} r \frac{(1 - \cos^2\theta)(\cos^2\theta + 1 - \cos^2\theta)}{(1 - \cos\theta)\{(1 + \cos\theta)\cos\theta\} + (1 - \cos\theta)^2\,(2 + \cos\theta)}$$

$$= \frac{3}{4} r \frac{1 - \cos^2\theta}{(1 - \cos\theta)\{(1 + \cos\theta)\cos\theta + (1 - \cos\theta)(2 + \cos\theta)\}}$$

$$= \frac{3}{4} r \frac{1 - \cos^2\theta}{(1 - \cos\theta)\,(\cos\theta + \cos^2\theta + 2 + \cos\theta - 2\cos\theta - \cos^2\theta)}$$

$$= \frac{3}{4} r \frac{1 - \cos^2\theta}{2(1 - \cos\theta)}$$

$$= \frac{3}{8} r\,(1 + \cos\theta) \qquad \cdots\cdots \; Eq.\ 158$$

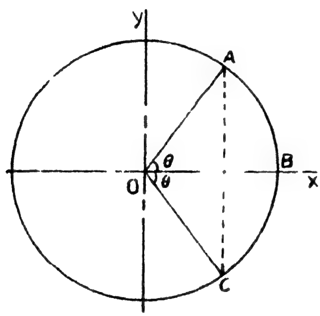


FIG. 203

Y and Z planes being the planes of symmetry,

$$\bar{y} = 0 \text{ and } \bar{z} = 0.$$

243. Stable, Unstable, and Neutral Equilibrium. If a body be suspended with a string, then, the point of suspension, S, and the centre of gravity, G, (Fig. 204A) must lie in the same vertical straight line with the string of suspension. The equilibrium maintained in

the body is called the *stable equilibrium.* In this case if the state of equilibrium be disturbed as in Fig. 204A₁, the weight of the body, acting at the centre of gravity, *G*, and the tension in the string of suspension, acting at *S*, being equal and parallel and opposite in direction will form a couple to bring the body back to the state of *stable equilibrium.* This couple is called a " righting couple ".

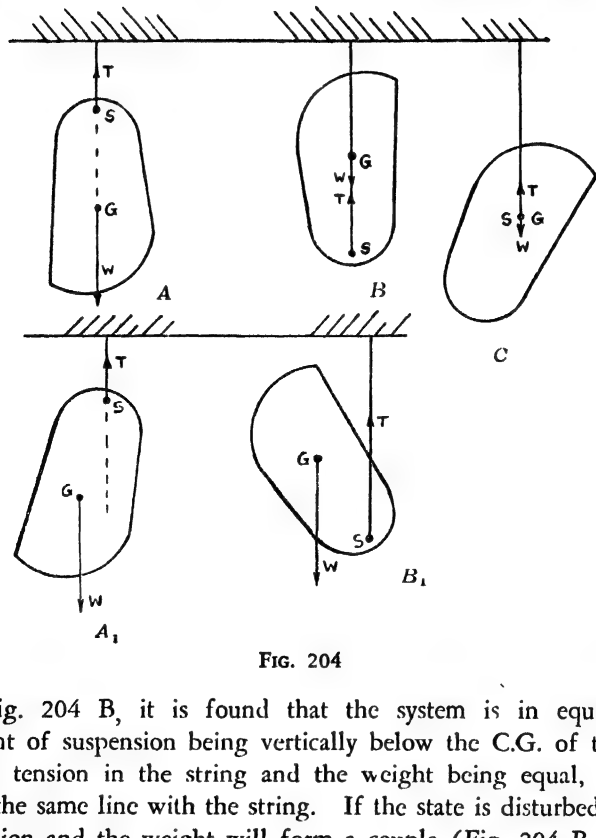


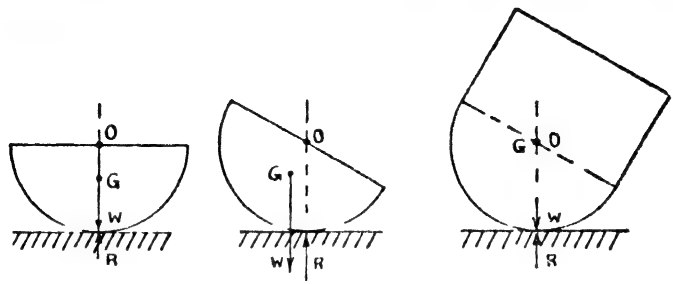
FIG. 204

In Fig. 204 B, it is found that the system is in equilibrium, the point of suspension being vertically below the C.G. of the body and the tension in the string and the weight being equal, opposite and in the same line with the string. If the state is disturbed a little, the tension and the weight will form a couple (Fig. 204 B_1) which will not help to bring the body back to its former state of equilibrium ; rather, it will rotate the body further to settle it to the state of stable equilibrium. This kind of equilibrium is called *unstable equilibrium* and the couple mentioned above is called an " upsetting couple ".

In Fig. 204 C is found a case of *neutral equilibrium.* The point of suspension and the centre of gravity coincide in this case. The

body may be placed in any position, the state of equilibrium will not be disturbed.

Illus. Ex. 120. *A body consists of a hemispherical part and cylindrical one, the diameter of which is equal to the diameter of the hemisphere and one of its plane surface coinciding with the plane surface of the hemisphere. The body is placed with its round surface on the horizontal plane of a table top. If the diameter of the hemisphere be 2 feet, determine the length of the cylindrical portion so that the body may attain the condition of neutral equilibrium.*



A B C

Fig. 205

The C.G. of a hemisphere, G, is at a distance of $\frac{3}{8}r$ from the centre along the radius at right angles to the plane surface, where r is the radius.

In the position as shown in Fig. 205 A, both the weight and the reaction, equal in magnitude but opposite in direction, will act along the vertical line through G. The body is in stable equilibrium.

If the hemisphere be tilted (Fig. 205 B), the weight will be acting through G downwards vertically and the reaction will be acting through the point of contact in the upward direction at right angles to the surface of the table top, and will pass through O, the centre of the hemisphere. These two forces will form a righting couple and will bring the body in the position of stable equilibrium.

Now, if a cylindrical portion be added as shown in Fig. 205 C, it is evident that the C.G. of the whole body will be shifted towards the plane surface of the hemisphere—the displacement will depend on the length of the cylindrical portion. There is a definite length for this cylindrical portion when G, the C.G. of the body will be shifted to O, the centre of the hemispherical body. In that condition both the weight and the reaction will be acting through O always and hence, the body may be placed in any position you like.

Therefore, the length of the cylindrical portion can be obtained from the relation,

$$\bar{x}, \text{ i.e., } r = \frac{\text{Vol. (hemisphere) } \frac{3}{8}r + \text{Vol. (cylinder)} \times (r + \frac{1}{2}l)}{\text{Vol. (hemisphere)} + \text{Vol. (cylinder)}}$$

Substituting the numerical values,

$$1 = \frac{\frac{2}{3}\pi\ 1^3 \times \frac{5}{8}.\ 1 + \pi\ 1^2\ l\ (1 + \frac{1}{2}l)}{\frac{2}{3}\pi + \pi.\ l}$$

From which, $l = .7071$ foot $= 8.485$ inches.

244. Base of a body, its C.G. and the Condition of Equilibrium. Take the case of a cylindrical body—its base resting on a horizontal surface. In Fig. 206 (1) the line of action of the weight of the

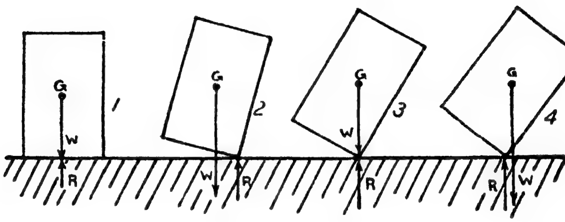


Fig. 206

body is passing through the centre of the base. The body is in equilibrium under the action of this weight and the reactions of the ground, the resultant of which will pass through the centre of the base and be in the same line with the weight. Now, if the body is turned over the edge (Fig. 206-2) as is shown in the diagram, a position is found where the line of action of the weight passes through some point in the base area at the left of the line of reaction that passes through the edge of the cylinder. The reaction and the weight form a righting couple to bring the body to its former state of equilibrium. In Fig. 206 diagram (3) shows the extreme position of not being turned over the edge and the line of action of the weight is yet within the limit of the base area, passing through the point of the edge where the reaction is acting. But, if the body is turned further, the line of action of the weight is shifted to the right of the point of the edge through which the reaction is acting and it is to be marked that the line goes outside the limit of the base area. In this case (diagram 4) the weight and the reaction will form an upsetting couple to topple the body over the edge. In each and every case of a body having a base area the previous phenomenon is observed. Thus it may be easily

said that till the line of action of the weight remains within the base area the body will not be turned over the edge.

By the term "Base Area" is meant the space covered by the stay of a body and not only by the actual area in contact with the surface. For example, if the case of a part of a cylindrical shall be considered as shown in the diagram (Fig. 207), the C.G. does not remain in the body itself, and, therefore, the line of action of the weight through G, the C.G. of the body, does not pass through any point in the surface of contact between the base and the ground. It passes through a point in the area, ABC, which may be said to be occupied by the 'stay' of the body.

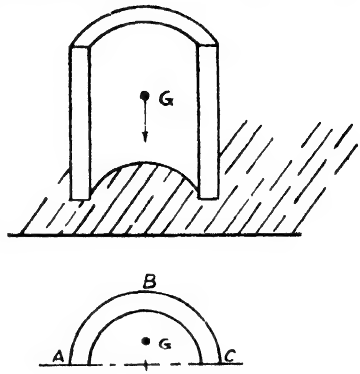


Fig. 207

Similar is the case with a cart or wagon. Till the vertical line through the C.G. of the body passes through a point in the area occupied by the wheels, the body will not be tilted over in spite of gradiant in the track.

245. Maximum Inclination of a plane on which a cylindrical body with its plane base in contact with the plane may rest without being tilted over. Let a cylindrical body rest on the inclined plane with its base on the plane (Fig. 208). The body will not be tilted over till the vertical line through the C.G. of the body is not shifted beyond the limit of the base area of the body, due to the inclination of the plane. The diagram is drawn in the extreme position of the body in equilibrium. If h be the altitude of the cylinder and r be the radius of the base area, then,

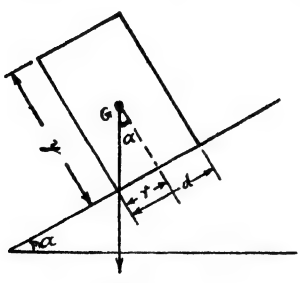


Fig. 208

$$\tan\alpha = \frac{r}{\frac{h}{2}} = \frac{2r}{h} = \frac{d}{h} \qquad \cdot Eq.\ 159$$

where r and d are the radius and diameter of the base circle respectively and α is the inclination of the plane.

246. Laboratory Methods of finding out the Centres of Gravity of Bodies.

There are various methods of which a few simple ones are stated below :

1. METHOD OF SUSPENSION.

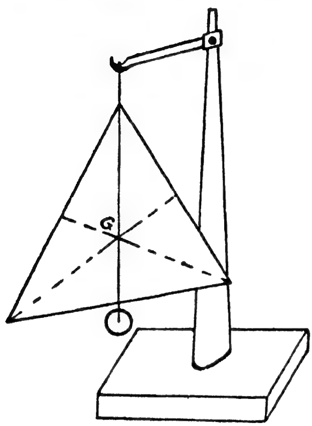


Fig. 209

(*a*) Lamina. It is evident that when a body is suspended from a point, the C.G. of the body must lie on the straight line coinciding with the thread of suspension.

From different points at the perimeter the lamina (Fig. 209) is suspended from a hook end as shown in Fig. 208. A small load is suspended from the same point. The straight lines are drawn on the surface of the lamina coinciding with the thread of suspension at different settings. The straight lines will be found to intersect at a common point. The C.G. must be on the plane of symmetry at the point where the perpendicular through the common point obtained above, cuts it.

(*b*) Frame Work :—Suppose the frame (Fig. 210) has three arms,

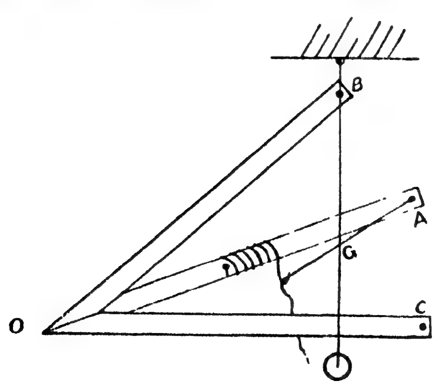


Fig. 210

OA, OB and OC. Adjust a flexible metallic wire to the arm, say, OA. First, suspend the frame from the point A and at the same time suspend a small metallic bob from a hook, the point of suspension of both the frame and the bob. Shape the flexible wire in such a way that a definite point on its surface is in touch with the string of suspension of the bob and mark it. Next, suspend the frame from the point B, as shown in the diagram (Fig. 210). Now, with the help of a second thread join A with the marked point in the flexible wire. It will be found that this string meets at a point with the string of suspension of the bob. That point indicates the position of the C.G. of the frame. The reason is obvious.

2. METHOD OF WEIGHING.

Two balances are adjusted at a distance such that the two ends of the given body rest on two knife edges at the two ends of the two balance arms respectively as shown in the diagram (Fig. 211).

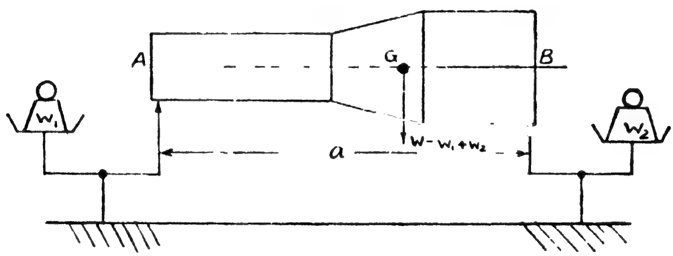


Fig. 211

The other two ends of the arms hold the pans for weights. Weights are placed on the pans to bring the balances in the state of equilibrium in horizontal position. These two weights are the reactions at the knife edges. The centre of these two forces gives the position of the C.G. of the piece with respect to the ends of the piece. The centre will be at such a distance from the ends that, $\frac{W_1}{W_2} = \frac{GB}{GA}$, where G is the centre of gravity of the piece.

247. To find out the Height of the C.G. of the bodies like Wagon, Cart, Locomotive, Cycle, etc. Take the case of a wagon. Let

the distance of the front and rear wheels be l and the position of the C.G. be G (Fig. 212). Suppose in a weighing machine the loads on

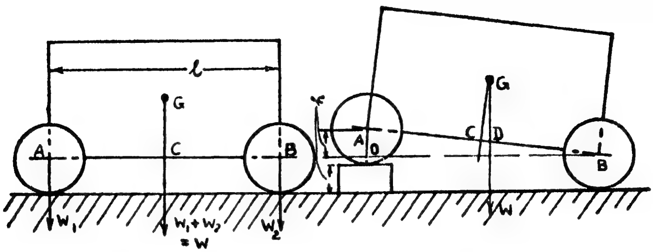


Fig 212

the front wheels and the rear wheels are indicated as W_1 and W_2 respectively. Then, the weight of the wagon is $W_1 + W_2 = W$ (say). Hence, the vertical line through G will cut a line joining the two axles, A and B, at right angles, say, at C in such a way that,

$$\frac{W_1}{W_2} = \frac{BC}{AC} \quad \text{or,} \quad \frac{W_1}{W_1 + W_2} = \frac{BC}{l}$$

$$\text{or, } BC = l \times \frac{W_1}{W_1 + W_2} = l \times \frac{W_1}{W}.$$

Now, raise the front wheels by h and weigh as before. If the weight on the front wheels is indicated to be W_x, from the similarity of triangles between the triangles, GCD and ABO,

$$\frac{GC}{CD} = \frac{BO}{AO} = \frac{\sqrt{l^2 - h^2}}{h},$$ or, GC, the height of the C.G. of the wagon over the centres of the wheels $= CD. \dfrac{\sqrt{l^2 - h^2}}{h}$. But, $CD = BC - BD$. Again, $BD = \dfrac{W_x}{W} l$

$$\therefore \quad GC = \left(l \times \frac{W_1}{W} - l \times \frac{W_x}{W}\right) \times \frac{\sqrt{l^2 - h^2}}{h}$$

$$= \frac{l\sqrt{l^2 - h^2}}{h} \times \frac{W_1 - W_x}{W}.$$

Therefore, if H be the height of C.G. from the track and r be the radius of the wheels, $H = r + \dfrac{l\sqrt{l^2 - h^2}}{h} \times \dfrac{W_1 - W_x}{W}$.

$$\cdots\cdots\cdots\cdots \textit{Eq. } 160$$

It is to be noted here that the diameters of the wheels have been taken equal, *i.e.*, the line AB in diagram A has been taken as horizontal.

If the centres of the front and rear wheels be not in the same level, the same procedure may be adopted to calculate the value of H. First, measure the body in its normal position and next measure the body by raising that end so that the line AB becomes horizontal.

Though this method is adopted to find out the C.G. of all kinds of wheeled bodies, such as, cycles, motor-cars, wagons, locomotives, etc. in cases of wagons and locomotives, for practical advantages wheels on one side of line are placed on a raised rail on the platform of the weighing machine so that the body is tilted sideways.

APPLICATION OF C.G.

248. Guldinus or Pappus Theorems.

THEOREM I. *The area of the surface of revolution of a plane curve, that lies wholly on one side of a straight line and in the same plane with it, generated by a complete revolution of the curve about that straight line is equal to the product of the length of the curve and the path described by its centre of gravity.*

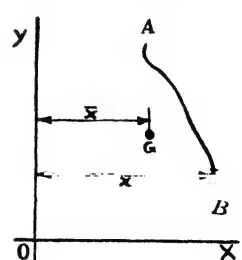


FIG. 213

Let the curve AB lie in the same plane with the Y-axis and wholly on one side of it. (Fig. 213). To find the area generated by one complete revolution of AB about the axis OY. Let G be the centre of gravity of the given plane curve, and its distance from OY be $\bar{x}$. Take an elementary length of the curve,

dl, whose distance from Y-axis is x. Then, for one complete revolution about the Y-axis this elementary length of the curve will generate an area $= 2\pi x\, dl$. Therefore, the area of the surface generated by the revolution of the whole curve $= 2\pi \int x\, dl$.

But, $\int x\, dl = x\ l$, where l is the length of the curve. Hence,

$2\pi \int x\, dl = 2\pi \bar{x}\, l$.

(*a*) If only a part of a revolution be made, *i.e.*, say, if the curve rotates through θ radians about the straight line, then, the area described by the curve is,

$$\frac{2\pi \bar{x} l \theta}{2\pi} = \theta\, x\, l$$

THEOREM 2. *The volume of the solid of revolution when of a plane area, lying wholly on one side of a straight line and in the same plane with it, generated by a complete revolution of the plane area about the straight line is equal to the product of the area and the path described by the centre of gravity of the area.*

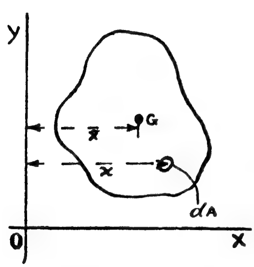


FIG. 214

Let the given plane area be in the same plane with the Y-axis and wholly on one side of it (Fig. 214). Also let G be the centre of gravity of the area, whose distance is x from the Y-axis, and let A be the measure of the area. Take an elementary area, dA, at a distance of x from the axis of revolution. Then, the volume generated by one complete revolution of the elementary area about the straight line is equal to $2\pi x\, dA$. Therefore, the total volume generated by one complete revolution of the area, $A = 2\pi \int x\, dA$. But, $\int x.\, dA = x\ A$

Hence, $2\pi \int x.\, dA = 2\pi\, \bar{x}\, A$

(*a*) If only a part of a revolution be made and if the angle turned through be θ radians, the volume generated $= \theta\, x A$

Illus. Ex. 121. ***Find the volume generated by one complete revolution of the area ABC (Fig. 215) about the axis XX.***

The C.G. of the area ABC is at a distance of $(2 + \frac{1}{3} \times 3)$, *i.e.*, 3 inches from the axis. The area of the triangular figure is equal to

$$\tfrac{1}{2} \times 4 \times 3 = 6 \text{ sq. inches.}$$

$\therefore$ the volume generated $= 2\pi \times 3 \times 6$

$= 121.04$ cubic inches.

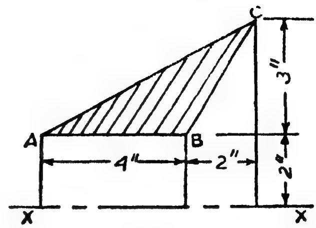


FIG. 215

Illus. Ex. 122. *What is the surface area of the volume obtained in the previous problem?*

The length of the straight line AC is found out from the relation, $AC^2 = 6^2 + 3^2 = 45$. From which, $AC = 6.708$ inches.

The distance of C.G. of AC from the axis of rotation $= 2 + \frac{3}{2} = 3.5$ inches.

Therefore, the area $= 2\pi \times 3.5 \times 6.708 = 147.5$ sq. inches.

249. Distributed Load and its Point of Application.

LOAD DIAGRAM. In Chapter VII we have dealt with concentrated loads. The point of application of each load is a single point. But if a load is spread over a surface area we call it a *Distributed Load.* Instances of this kind of load is found in beams supporting the roof of a building, retaining walls of dams, resisting the water pressure, etc. The distributed load can be represented by diagrams, called, *Load Diagrams.* Take the case of a beam. In case of ordinary beams of a building the load is taken as uniformly distributed. The weight of the roof divided by the number of beams is the load on each beam. This load divided by the length of the beam, *i.e.*, the load per unit length of the beam, is the magnitude of the uniform distribution of the load along its length. It is also called the *Intensity of the Load.* It is evident that the portion of the roof supported by the beam is a rectangular piece of block of uniform thickness. Therefore, if W be the total load, $W = l \times b \times t \times w$, where l is the length of the block, *i.e.*, of the beam, b is the width, t the thickness and w the density of the load material.

$$\text{Now, } w.t.b = \frac{W}{l}\text{, or, } t\,(bw) = \frac{W}{l}$$

Therefore, w and b being constant quantities, $\frac{W}{l}$ can be represented

by t if the scale be properly chosen. In other words, we can say that if AB (Fig. 216) be the beam, then, the load on it can be represented by the rectangular area $ABCD$. If h be the height of this area, *i.e.*, $h = BC$ or AD, then, h will represent $w.b.t$ in some definite scale. The area $ABCD = l \times h = l \times b \times t \times w = W$.

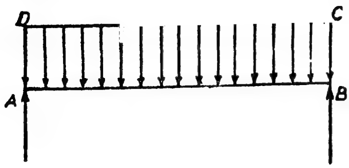


FIG. 216

At any section of the beam the intensity of load is the depth of the rectangular area at that section. The area $ABCD$ representing the load is called the *Load Diagram*. The line DC is called the *Load Intensity Curve.*

It is to be marked that the lines of action of the weights on different portions of lengths of the beam will be on a vertical plane passing through the centres of gravity of the transverse sections of the beam. Therefore, the weights can be taken as co-planer parallel forces. The beam is represented by a straight line passing through the centres of gravity, which is called the axis of the beam.

In case of a retaining wall of a dam we know that with the depth of the water the pressure increases and it is proportional to the depth. Therefore, if by horizontal vectors (Fig. 217) the intensity of pressure at different points along the height of the wall be represented, then, the vector lengths being proportional to the depth of water, the line

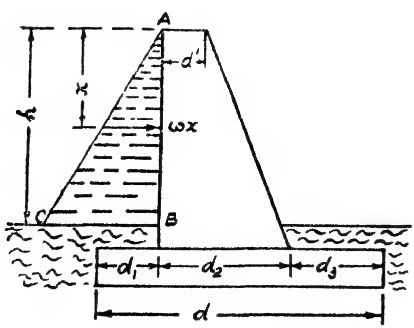


FIG. 217

joining the ends of the vector lengths will be a straight line as shown —AC. This line is the pressure intensity curve or the load intensity

curve and the triangle ABC is the load diagram. At any depth x the intensity is $w.x$, where w is the weight of water per unit volume and the vector at that point represents the value of $w.x$. This is an instance of uniformly varying distributed load, varying from Zero at the top to Maximum at the bottom.

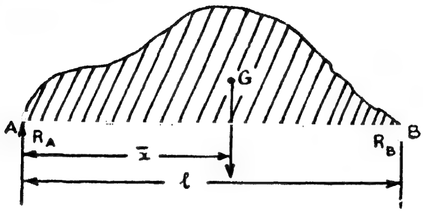


FIG. 218

There are cases where the distribution of load is more complicated and the load intensity curve is an irregular curve as shown in Fig. 218.

CENTRE OF DISTRIBUTED LOAD

(1) When the load is uniformly distributed as shown in Fig. 219.

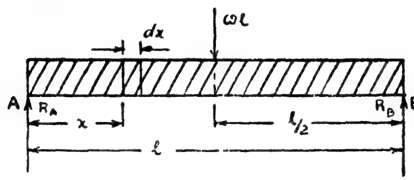


FIG. 219

$$\int_0^l h\,dx\,x = W x, \quad \textit{where } \bar{x} \text{ is the distance of the centre,}$$

$$i.e., \int_0^l b.\,t.\,w.\,dx\,x = W x$$

$$\text{or,}\quad \bar{x} = \frac{b.\,t.\,w \int_0^l x\,dx}{W} = \frac{bt\,w\,l^2}{2W} = \frac{w\,l}{2\,W} = \frac{l}{2},$$

that is, the C.G. of the load is at the middle of the beam length.

(2) When the load is uniformly varying distributed load as in figure 118.

In this connection we can go back to the Illustrated Ex. 103. In that case if the intensity of stress is substituted by the load intensity, the result will remain the same. It is evident from the result that the centre is the centre of gravity of the triangular area representing the load diagram. (3) In case of irregularly varying load intensity curve (Fig. 218) if at a distance x from the left hand end the intensity is represented by a height h, then, on a differential length dx the load is $h.\,dx$. Hence,

$$\int_A^B h.\,dx.\,x. = W\bar{x}, \text{ or, } \bar{x} = \frac{\int_A^B h.\,dx.\,x}{W},$$

which is in the form of the equation to find out C.G.

250. Reaction on Beams. The weight of a body is the resultant of the weights of the elementary particles composing the body and the resultant acts through the C.G. of the body. It is found by computation that if a single force, equal to the weight of the body, acts at the C.G. and if the body is considered weightless, then, it will produce the same effect on the support as the weights of the elementary particles would produce collectively.

For example, take the case of a uniformly loaded beam simply supported at two ends. The words 'uniformly loaded' means that the load per unit length of the beam is constant. Let the load per unit length of the beam be w (including the weight of the beam itself). Let AB be the beam and let its length be l (Fig. 219). To find the reactions, R_A and R_B, at the supports at A and B respectively. Because the system is in equilibrium under the action of loads and the reactions at the supports, the sum of the moments of all the forces acting on the beam about any point in the plane is equal to Zero. Take an elementary length dx at a distance x from the end A (say), then, taking moments about A, $\int_0^l w\ dx\ x = R_B\ l$

$$\text{or, } R_B = \frac{\int_0^l w\,dx\,.\,x}{l} = \frac{w\,l^2}{2\,l} = \frac{w\,l}{2} \qquad \cdots\cdots\cdots\text{Eq. } 161$$

The same result would be obtained if a concentrated load of an amount equal to wl be assumed to act at the middle point of the

beam, (which is, of course, the C.G. of the beam and the load), instead of uniformly distributed load, because, on taking moment about A,

$$wl.\frac{l}{2} = R_B.l, \quad \text{or, } R_B = \frac{wl}{2}.$$

Similarly, in cases of beams where loads are not uniformly distributed the reactions can be easily found out in the same way if the position of the centre of gravity of the load be known (Fig 218).

$R_B = \frac{W\bar{x}}{l}$, where W is the total load and $\bar{x}$ is the distance of C.G. of the load from A.

and $R_A = W - R_B$

$$= W - \frac{W\bar{x}}{l} = \frac{W(l - \bar{x})}{l}.$$

Illus. Ex. 122. *A sluice gate when closed, has water on both the sides—on one side the height of the level is 15 feet and on the other side 10 feet from the bed. Determine the intensity of pressure due to reaction at the guides, A and B, (Fig. 220).*

Let W_1 and W_2 be the loads per foot width of the gate on the sides of the higher and lower levels respectively.

Then, $W_1 = 62.5 \times \frac{15}{2} \times 15$

$= 7033$ lbs.

and $W_2 = 62.5 \times \frac{10}{2} \times 10$

$= 3125$ lbs.

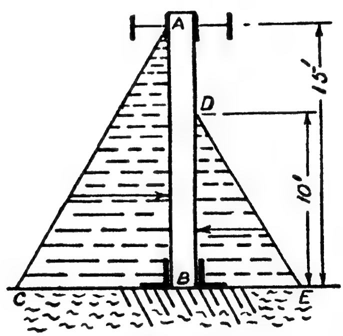


FIG. 220

Taking moments of all the four forces acting in the system—(Intensities of the reactions R_A and R_B at A and B respectively, W_1 and W_2 at centres of gravity of the two triangular areas ACB and DEB respectively) about A,

$$R_B \times 15 + 3125\,(15 - \tfrac{1}{3} \times 10) - 7033 \times \tfrac{2}{3} \times 15 = 0$$

From which $R_B = +\ 2258$ lbs. per sq. ft. Plus sign indicates that the direction is from E towards B.

Again, $\Sigma H = 0$

Therefore, $7033 - 3125 - 2258 + R_A = 0$

From which $R_A = -1650$ lbs. Here, minus sign indicates that the direction is in the same sense with the previous one, because the direction of W_1 has been taken in the positive sense.

The value of R_A can also be found out by the principle of moments.

251. Centrifugal Force due to uniform rotation of a body about an axis. Suppose a body (Fig. 221) rotates about the axis OZ of the three rectangular co-ordinate axes. Take an elementary volume dV of the body at a distance r from the Z-axis and let x and y be the distances of the volume from the X and Y planes respectively. Let w be the weight of the body per unit volume. Then, the centrifugal force on the elementary volume is equal to $\frac{w}{g} dV.\omega^2.r$, where ω is the constant angular velocity of the body about the axis of rotation. Resolving this force parallel to Y and X planes respectively, the components are, $\frac{w}{g} dV.\omega^2.r\cos\theta$ and $\frac{w}{g} dV.\omega^2.r\sin\theta$, where θ is the angle made by the centrifugal force with the Y-plane. But, $r\cos\theta = x$ and $r\sin\theta = y$. Therefore, the components are, $\frac{w}{g} dV.\omega^2.x$ and $\frac{w}{g} dV.\omega^2.y$. Hence, the total centrifugal force on the whole mass is equal to $\int \frac{w}{g} dV.\,\omega^2.\,r$

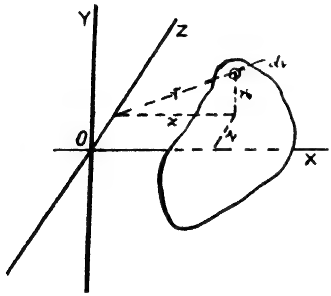


FIG 221

$$= \sqrt{\left(\int \frac{w}{g}\, dV.\,\omega^2.\,x\right)^2 + \left(\int \frac{w}{g}\, dV.\,\omega^2.\,y\right)^2}$$

$$\text{or, } \frac{w}{g}\,\omega^2 \int dV.\,r = \frac{w}{g}\,\omega^2 \sqrt{(\int dV.x)^2 + (\int dV.y)^2}$$

$$= \frac{w}{g}\,\omega^2 \sqrt{V^2.\,\bar{x}^2 + V^2.\,\bar{y}^2}$$

$$= \frac{w}{g}\,V\omega^2 \sqrt{\bar{x}^2 + \bar{y}^2} = \frac{W}{g}\,\omega^2 \sqrt{\bar{x}^2 + \bar{y}^2}$$

But, $\sqrt{\bar{x}^2 + \bar{y}^2}$ is the distance of the C.G. of the body from the Z-axis. Let this distance be represented by $\bar{r}$. Then,

$$P = \frac{W}{g} \omega^2 \bar{r} \qquad \cdots\cdots\cdots\cdots Eq.\ 162$$

where P is the centrifugal force.

252. Work done in lifting a mass. In raising a pitcase or filling an overhead tank with water it is evident that different portions of the mass raised will undergo different displacements. Let a mass of weight W be raised through a height. To find out the work done.

Take the case of raising water from a reservoir to an overhead tank (Fig. 222). Let an elementary volume of water in the reservoir, dV, be at a height of y_1 from a reference level, also let the C.G. of the volume of water to be raised, be at a height of y' from the same level. Then,

$$W . y' = w \int dV . y_1,$$

where w is the weight of water per unit volume.

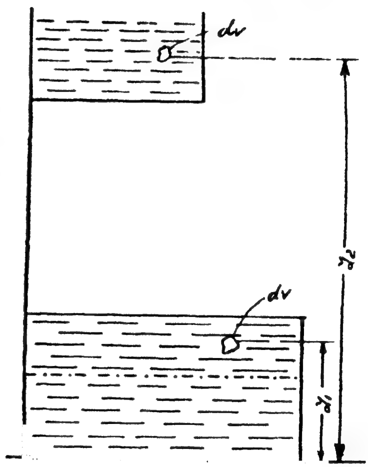


FIG. 222

Now, suppose the mass is raised to a desired level, and the elementary volume rises thereby to a height y_2 from the reference level. Let the distance of C.G. of the total mass raised, in its new position, be y'' from the same level. Then,

$$W . y'' = w \int dV . y_2$$

The work done in raising the elementary volume,

$$= dV . w\,(y_2 - y_1)$$

Hence, the work done in raising the total mass,

$$Q = w \int dV\,(y_2 - y_1)$$

$$= w \int dV . y_2 - w \int dV . y_1$$

$$= W . y'' - W . y' = W (y'' - y') \qquad \cdots\cdots\cdots\cdots Eq.\ 163$$

But, $(y'' - y')$ is the distance through which the C.G. of the mass has been shifted. Hence, the work done in raising a mass is equal to the weight of the mass multiplied by the distance through which the C.G. of the mass is raised.

Illus. Ex. 123. *A chain l feet long is suspended from a drum of a lifting machine. The weight per foot length of the chain is w lbs. Find the work done in winding the total length of the chain round the surface of the drum. What is the work done in winding half of the chain?*

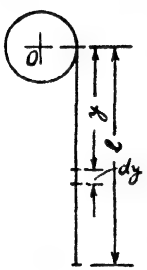


FIG. 223

Take an elementary length, dy, of the chain (Fig. 223) at a distance of y feet from the point of suspension. Then, the work done in lifting the elementary mass to the point of suspension is equal to $dy.w.y$. Therefore, the work done in raising the full length of the chain,

$$Q = w \int_0^l y . dy = \frac{w\, l^2}{2} = \frac{W\, l}{2} \text{ ft. lbs.}$$

Alternative Method

The C.G. of the chain, when suspended, is at a distance of $\frac{l}{2}$ feet from the point of suspension. When the chain is totally wound up, the C G. of the mass rises to the level of the point of suspension. Therefore,

$$Q = W . \frac{l}{2} \text{ ft. lbs.} \qquad (Eq.\ 163)$$

When only half of the length of the chain is wound up,

$$Q = w \int_{\frac{l}{2}}^{l} y . dy = \tfrac{3}{8} W\, l$$

Illus. Ex. 124. *A 100 feet chain is suspended round a frictionless pulley fitted at a height of 70 feet from the ground level. One end of the chain touches the ground. For advantage of work the two ends of the chain are brought to a level difference of 6 feet only. Find the work necessary to perform the job. The weight of the chain is 10 lbs. per foot.*

Neglecting the consideration of the curvature of the pulley, which is negligible in comparison with the length of the chain, the distance of the C.G. for the first position is equal to,

$$\frac{(70 \times 10 \times 35) + (30 \times 10 \times 15)}{1000} = \frac{29000}{1000} = 29 \text{ ft.}$$

The distance of the C.G. for the second position,

$$= \frac{(53 \times 10 \times 26.5) + (47 \times 10 \times 23.5)}{1000} = \frac{25090}{1000} = 25.09 \text{ ft.}$$

Therefore, the distance through which the C.G. is raised $= (29 - 25.09)$ $= 3.91$ ft.

Hence, the work done, $Q = 3.91 \times 1000 = 3910$ ft. lbs.

253. D'Alembert's Principle.

REACTION OF THE TRACK ON THE DRIVING WHEEL OF A CAR, CYCLE, LOCOMOTIVE ENGINE, ETC.

The frictional resistance between the wheels and the track which is equal to the force that can be applied on the driving wheel by the driving agent to move a vehicle, acts in the direction of the motion of the vehicle.

It is to be noted that in case of sliding bodies the frictional force acts in a direction opposite to the motions of the bodies, whereas, in case of rolling bodies, if it be a power wheel (such as the driving wheels of railway engines, motor car, etc.), it acts in the same direction with the motions, and if it be an idle wheel (such as wheels of a wagon, cart, etc.), it acts against the motions.

The statical conditions of equilibrium $\Sigma V = 0$, $\Sigma H = 0$ and $\Sigma M = 0$, are sufficient to solve almost all general statical problems. With the help of these conditions some problems of Dynamics can also be solved easily. In order to do that a body in motion is made to be conceived in a state under which all the external forces acting on it may be considered to form a balanced system. This condition is called the *Dynamical* condition of equilibrium. D'Alembert's Principle explains the way by which we can attain to that condition. It is explained below.

D'Alembert's Principle says, "If forces equal to the effective forces but acting in exactly opposite directions were applied at each point of a system, these would be in equilibrium with the impressed forces."

Here, the effective forces referred to above are the accelerating forces. The principle may be modified in the following way :

Suppose a body A (Fig. 224) moves in a direction shown by the arrow-head with an acceleration f. Then, each of the elementary particles composing the body will move with the same acceleration. If P_1, P_2, P_3, etc., are the accelerating forces acting on the elementary particles, m_1, m_2, m_3, etc., respectively,

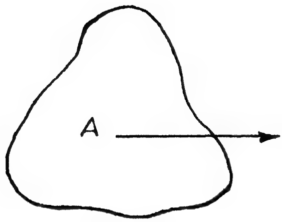


FIG. 224

$$P_1 = m_1 . f$$
$$P_2 = m_2 . f$$
$$P_3 = m_3 . f$$
$$\cdots\cdots\cdots\cdot$$
$$\cdots\cdots\cdots\cdot$$

Now, f being a constant quantity, forces are proportional to the masses and they are acting in the same direction, *i.e.*, all of them are parallel like forces. Therefore, the centre of these parallel forces is the centre of gravity of the body A. Thus, the resultant of the accelerating forces acting on the elementary masses, *i.e.*, the effective forces, will act at the centre of gravity of the body. Hence, if an equal and opposite force can be introduced at the centre of gravity of the body, then, all the forces acting on the body will form a balanced system. The force introduced to form a balanced system is called *reversed effective force.*

Thus, if we know the acceleration of a body in motion, then, directly with the relation, $P = m f$, we can determine the accelerating or effective force acting on the body and we can just say that if a force, equal and opposite to the accelerating force of a body, be introduced at the centre of gravity of the body, then, this force with impressed force on the body will form a balanced system.

Take the case of a motor car (Fig. 225). The five forces—$\frac{w}{g} f$, w, R_1, R_2 and F—will be in equilibrium (the car has been taken to have a front-wheel drive). Of these R_1, R_2 and F are generally unknown. If, however, the position of C.G. of the car is known, *i.e.*, if H and l_1 be known, then, taking moments of all these forces about some advantageous point (here, in this case taking the moments about the point of contact of the rear wheel with the track) an equation with only one unknown quantity can be obtained,

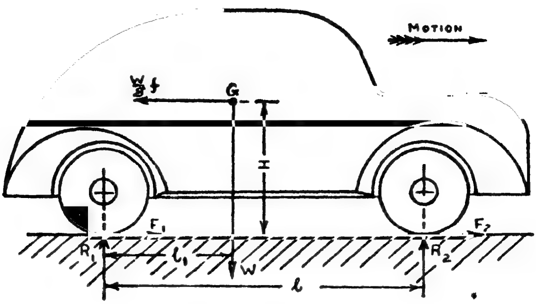


Fig. 225

$$w\,l_1 - R_1\,l - \frac{w}{g}\,f\,H = 0, \text{ from which}$$

$$R_1 = \frac{w}{l}\left(l_1 - \frac{f\,H}{g}\right) = R' - \frac{w\,f\,H}{g\,l} \quad \cdots\cdots\cdots\cdots \quad Eq.\ 164$$

where R' represents the reaction on the front wheel at the state of rest or of uniform motion, which is equal to $w\,\frac{l_1}{l}$.

Again, if the car has a rear wheel drive, taking moments about the point of contact of the front wheel with the track,

$$R_2\,l = \frac{w}{g}\,f\,H + w\,(l - l_1)$$

$$\text{or,} \quad R_2 = \frac{w}{g\,l}\,f\,H + \frac{w\,(l - l_1)}{l}$$

$$= R'' + \frac{w\,f\,H}{g\,l} \qquad Eq.\ 165$$

where R'' represents the reaction on the rear wheel at the state of rest or of uniform motion and is equal to $\frac{w\,(l - l_1)}{l}$

Now, compare the two cases of front wheel drive and the rear wheel drive. It is to be found that in case of front wheel drive the reaction decreases, whereas, in case of rear wheel drive the reaction increases and the quantity of change is the same in both the cases. But, it is to be noted that the sum of the two reactions on the front and the rear wheels, is always equal to w, the weight of the car.

Illus. Ex. 125. *In a motor car the C.G. is 3 feet above the track level and 4 inches backward from the middle of the axle distance, which is 9 ft. 4 ins. The gross weight of the car is 1 ton and the car has a rear wheel drive. (1) If the car runs with an acc. of 3 feet per second per second determine the reaction on the driving wheel. (2) If the coefficient of adhesion between the wheels and the track be .4, find the maximum acceleration that can be produced. (3) What is the horse-power of the engine in the first case after 4 seconds of drive?*

(1) $R'' = 2240 \times \dfrac{112 - 50}{112} = 1200$ lbs.

Therefore, $R_2 = 1200 + \dfrac{2240 \times 3 \times 3 \times 12}{32.2 \times 112}$

$= 1200 + 67.08 = 1267.08$ lbs.

Or,

Taking moments about the point of contact of the front wheel with the track, $R_2 \times 9\frac{1}{3} = \dfrac{2240 \times 3 \times 3}{32.2} + 2240\left(9\frac{1}{3} - 4\frac{1}{3}\right)$

From which, $R_2 = 1267.08$ lbs.

(2) $F = \dfrac{w}{g} f$ and $R_2 = \dfrac{F}{\mu}$ and $R_2 = R'' + \dfrac{w f H}{g l}$

Therefore, $R'' + \dfrac{w f H}{g l} = \dfrac{w f}{g \mu}$

$$\text{or, } f\left(\frac{w}{g\mu} - \frac{w H}{g l}\right) = R''$$

Now, substituting the numerical values of the notations,

$$f\left(\frac{2240}{.4 \times 32.2} - \frac{2240 \times 3}{32.2 \times 9\frac{1}{3}}\right) = 1200$$

From which, $f = 7.913$ feet per second per second.

(*Bearing friction loss has been neglected*)

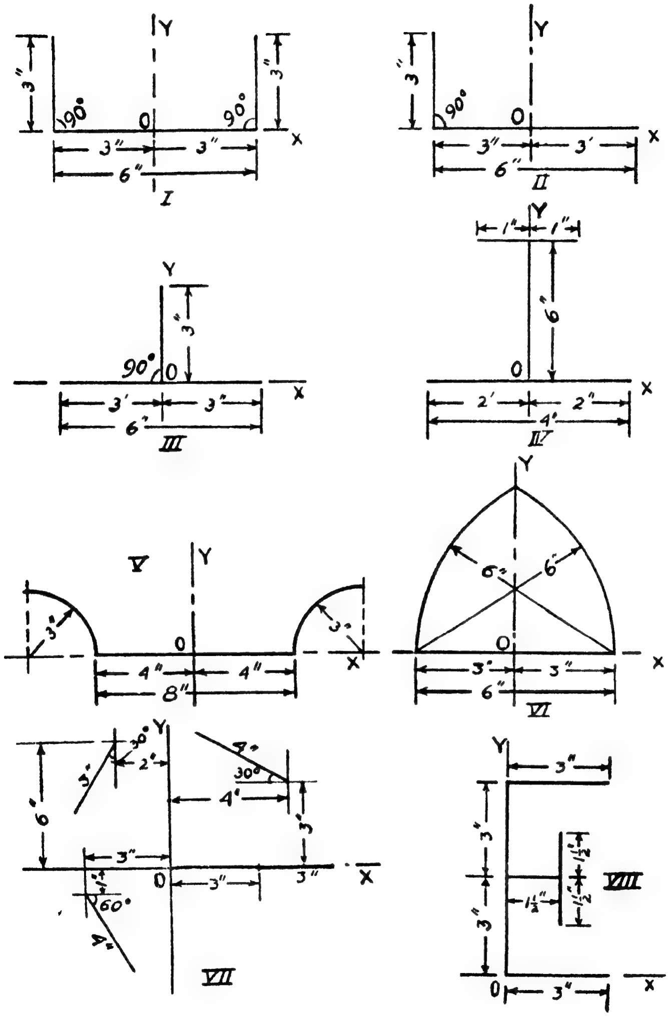
(3) Velocity after 4 seconds $= 4 \times 3 = 12$ feet per second.

$F = .4 \times 1267.08 = 506.83$ lbs.

Therefore, $H.P. = \dfrac{506.83 \times 12}{550} = 11.06$

PROBLEMS

Locate the position of C G in each case of the systems of lines given in the block below (Fig. 226--233), with respect to X and Y axes.



FIGS. 226-233

277. Fig. I. *Ans* $\bar{x} = 0$

$\bar{y} = .75$ in.

278. Fig. II. *Ans.* $x = -1$ in.

$\bar{y} = .5$ in.

279. Fig. III. *Ans.* $\bar{x} = 0$

$y = .5$ in.

280. Fig. IV. *Ans.* $\bar{x} = 0$

$y = 2.5$ ins.

281. Fig. V. *Ans.* $\bar{x} = 0$

$\bar{y} = 1.033$ ins.

282. Fig. VII. *Ans.* $x = .452$ in.

$\bar{y} = 1.36$ ins.

283. The shape of the Figure VIII is formed with the help of a fine flexible wire 21 inches long. Find the position of the C.G. of the wire in its new shape.

Ans. $\bar{x} = .96$ in.

$y = 3$ ins.

Determine the position of C.G. in each case of the shaded area given below, with respect to X and Y axes.

284. Fig. 234.

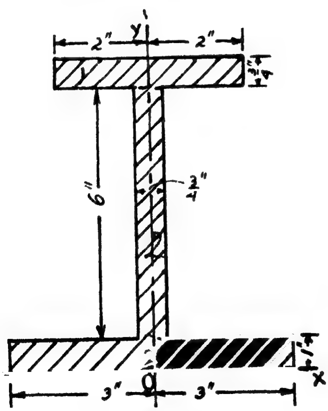


Fig. 234

Ans. $\bar{x} = 0,$

$\bar{y} = 3.194$ ins.

285. Fig. 235.

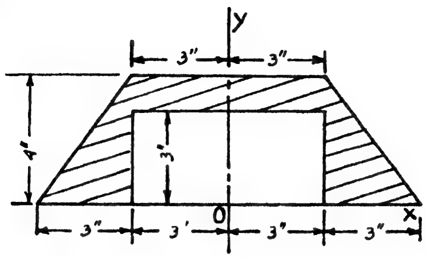


FIG 235

Ans. $\bar{x} = 0,$

$\bar{y} = 1.6$ ins.

286. Fig 236.

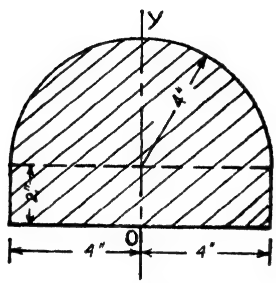


FIG. 236

Ans. $\bar{x} = 0,$

$\bar{y} = 2.3$ ins.

287. Fig. 237.

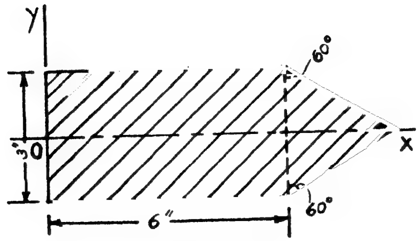


FIG. 237

Ans. $\bar{x} = 3.69$ ins.,

$\bar{y} = 0$

288. Fig. 238.

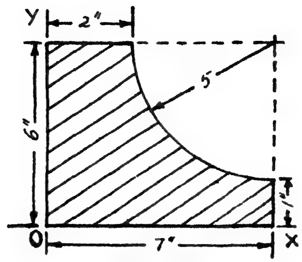


FIG. 238

Ans. $\bar{x} = 2.29$ ins.,
$\bar{y} = 2.22$ ins.

289. Fig. 239.

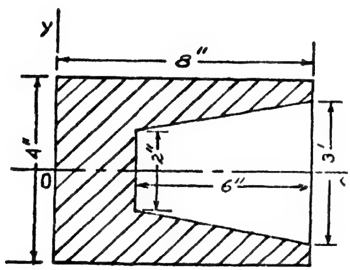


FIG. 239

Ans. $x = 3.1$ ins.,
$\bar{y} = 0$

290. Fig. 240.

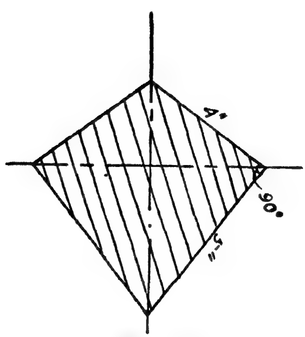


FIG. 240

Ans. $\bar{x} = 0$,
$\bar{y} = .153$ in.

291. Fig. 241.

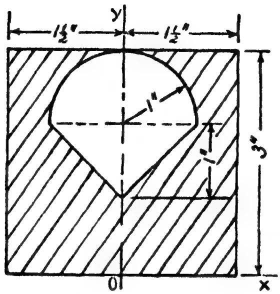


FIG. 241

Ans. $x = 0$,

$\bar{y} = 1.25$ ins.

292. Fig. 242.

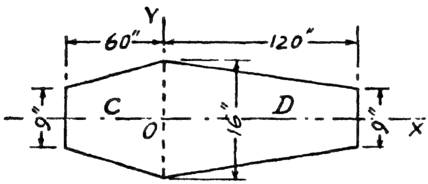


FIG. 242

Ans. $\bar{x} = 0$,

$\bar{y} = 8.44$ ins.

293. Find the distance of the C.G. of the area enclosed by two concentric semicircles and the common diameter as shown in Fig. 243 from the X-axis.

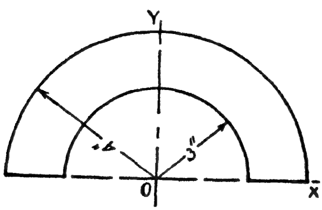


FIG. 243

Ans. $\bar{y} = 2.3$ ins.

294. Prove that if from a triangular area a similar area having common medians be taken off the C G remains unaltered

295. Determine the position of the C.G. of the area given in Fig. 244, from the X-axis.

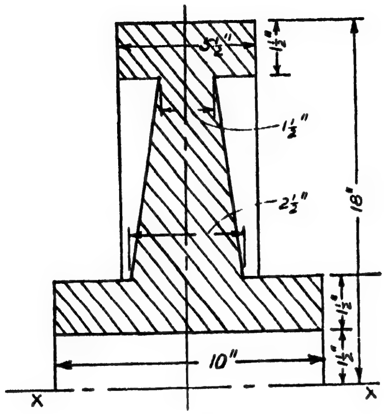


FIG 244

Ans $y = 8\ 44$ inches

296. Determine the position of the C G of a body which is shown in section in Fig 234—two circular discs connected by a cylindrical part, all the three are co axially placed—with respect to three rectangular co ordinate axes passing through O, X axis being on the plane of the base

Ans $x = 0$, $y = 2\ 335$ ins, $z = 0$

297. Fig 235 represents the section of a frustum of a cone having a cylindrical portion with common axis removed Locate the positoin of the C G of the body with respect to three axes passing through O, the centre of the base area—X-axis lies on the plane of the base area

Ans $x = 0$, $\bar{y} = 1\ 6$ ins, $z = 0$

298. Fig 236 is the section of a body having a circular disc as base with a hemisphere, the plane surface of which coincides with the plane surface of the disc, placed on it Determine the position of the C G of the body with respect to three axes passing through O the centre of the bottom face of the disc—X-axis lying on the bottom face of the disc

Ans $x = 0$, $y = 2\ 36''$, $\bar{z} = 0$

299. Fig 237 is the sectional view of a composite body with two parts—one cylindrical and the other a conical, the base of the cone coincides with a plane face of the cylinder having common axis.

Ans. $\bar{x} = 0$, $\bar{y} = 3.46$ ins., $z = 0$

300. Fig. 239 is the sectional view of a body cylindrical in shape having a portion removed, the form of which is a frustum of a cone as shown. The removed portion has the same axis with the main body.

Ans. $\bar{x} = 0$, y = , $z = 0$

301. Fig. 240 represents the sectional view of a body formed by two conical portions, having common base as shown. Determine the distance of the C.G. of the body from the right hand appex of the body. *Ans.* 3.34 ins.

302. A rectangular block of material 2.5 feet long, 2 feet broad and 4 feet high, is submerged into the water of a reservoir, $10' \times 10' \times 10'$, and the water is full to the brim. The weight of the material of the block weighs 250 lbs. per cu. ft.

(1) Locate the position of the C.G. of the water only in the reservoir from the water surface.

(2) Find the depth of the C.G. of the total mass in the reservoir from the surface of the water.

(3) Determine the distance of the C.G. of the combined masses when the block is raised till it clears the surface of water from the bottom of the reservoir.

Ans. 4.94 ft., 5.17 ft., 5.42 ft.

303. A horizontal beam 20 feet long is simply supported at the two ends. If the intensity of the load on the beam varies uniformly from $\frac{1}{4}$ ton per foot run at one end to 4 tons per foot run at the other, including the weight of the beam, find the reactions at the two ends.

Ans. 27.5 tons, 15 tons.

304. A beam simply rests horizontally on supports 15 feet apart. The load increases at uniform rate from nothing at one end. If the total load be 10 tons, find the reactions at the ends neglecting the weight of the beam.

Ans. $6\frac{2}{3}$ tons ; $3\frac{1}{3}$ tons.

305. A horizontal beam *AB* rests simply on supports *A* and *B*. A load *W* is spread over it—firstly, in a way so that the intensity varies uniformly from 0 ton per foot run at *A* to 8 tons per foot run at *B*, secondly, in a way so that the intensity changes uniformly from 2 tons per foot run at *A* to 6 tons per foot run at *B*. It is found that the reaction at *A* increases by 8 tons due to the second arrangement. Find the load *W* and the length of the beam. *Ans.* $W = 8$ tons, $L = 8$ feet.

306. A horizontal beam, 20 feet long, simply rests on two supports at the two ends. A uniformly distributed load of 2 tons acts for 12 feet of, the beam from one end and then, the load uniformly decreases to zero at the other end. Find the C.G. of the loading system and also the reactions of the two supports. *Ans.* 8.5 ft. from the end of 2 tons intensity, 18.4 tons, 13.6 tons.

307. What uniform wind pressure per sq. ft. can be resisted without being tilted over, by a portable back screen in a cricket field, the description of which is given below?

Total weight of the frame with fittings = 1 ton.
Screen, height = 12 feet, length = 25 feet.
Length of each foot frame = 8 feet
The position of the screen is at the middle of the stays.

Ans. Approximately 5 lbs. per sq. ft.

308. A beam AB, 25 feet long, rests horizontally on two supports C and D, which are 5 and 15 feet respectively from the end A. The load on the beam is uniformly distributed over its length and combining with the weight of the beam the load weighs 25 tons. Three concentrated loads of 1, 2 and 3 tons are applied at distances of 2, 12 and 20 feet respectively from the end A. Determine the reactions of the supports at C and D.

Ans. At C, 6.65 tons and at D, 24.35 tons.

309. A brick wall 100 feet long is raised to stop the water on one side of it to enter into the other side. If the water level be at a height of 10 feet from the soil and if the foundation of the wall be extended up to 5 feet into the soil, find the reversed couple produced by the earth resistance so that the wall is not tilted over due to water pressure. *Ans.* 272916.25 lb. ft.

310. In a retaining wall (Fig. 217) AB is 15 feet, $d' = 2$ feet, $d_2 = 4$ feet, $d_3 = 2$ feet and the height of point B is 3 feet from the bottom of the foundation slab. If the water level is 10 feet from the ground and if the pressure intensity of the earth on the base slab varies uniformly from zero at one end to maximum at the other, compute the maximum pressure intensity on the slab due to the resistance of the earth.

311. What is the area of the surface of a frustum of a cone 10 inches high, if the diameters of the plane surfaces at the two ends are 12 and 6 inches respectively? *Ans.* 295 sq. ins.

312. Find the volume removed if 6 grooves of cross-section $\frac{3}{8} \times \frac{3}{8}$ in. are cut round the surface of a piston, 6 inches in diameter, for the fitting of the springs.

313. Estimate the quantity of metal required for a cast-iron disc-fly-wheel which is shown by half in section in the diagram of Fig. 244. C. I. weighs .26 lb. per cu. in. What is the distance of the C.G. of the area given from the X-axis? *Ans.* 692 lbs., 8.44 inches.

314. A cable, 100 feet long, is suspended from a winding machine. If the weight of the cable is 8 lbs. per foot run, find the work done (1) to wind it completely and (2) to wind it by 75 feet only. Neglect the friction loss. *Ans.* 40000 ft. lbs.
37500 ft. lbs.

315. If a cage weighing 15 cwt. be lifted by the cable in the previous problem, find the work done under the same conditions to raise the maximum height. If the cage is lifted in 1.5 minutes, find with what average horse-power the machine is working if the friction loss is 50 p.c.

Ans. 58000 ft. lbs., 2.34 *H.P.*

316. A chain, 80 feet long, is suspended about a pulley holding two buckets at the two ends. The chain weighs 10 lbs. per foot length. Initially the difference between the levels of the two buckets is 30 feet. Find the work done in bringing the two buckets at a level difference of 3 feet only. Neglect the friction of the pulley and the weight of the buckets.

317. A horizontal cylindrical boiler shell of 5 feet internal diameter and 15 feet length is half filled with water from a reservoir, $15' \times 10'$ surface area. Initially the water surface is 15 feet below the shell. Find the work done. *Ans.* 157000 ft. lbs.

318. Two rectangular tanks, $10' \times 5'$ and $5' \times 4'$ cross-sectional area, are connected at the bottom by a narrow pipe to permit a flow of the fluid from one to the other. In empty condition the pipe is closed and the bigger tank is filled with 350 cu. ft. of water. Then the union between the two tanks is established and the water flows slowly to the empty tank. What is the work done when the flow of water ceases and the whole system comes to a stability? *Ans.* 21875 ft. lbs.

319. A hollow cast-iron cylindrical column—outer diameter 15 inches and the inner diameter 12 inches and 20 feet long—is lying on the ground. Compute the work done in raising it against one of the ends to a vertical position. Cast-iron weighs .26 lb. per cu. in. *Ans.* 36380 ft. lbs.

320. 5400 pieces of brick are loaded on a railway wagon weighing 10000 lbs. being uniformly spread over the floor. The weight of each piece of brick is 10 lbs. The gauge of the line is 4 ft. $4\frac{1}{2}$ ins. It is found that when a pair of wheels on one side is placed on a weigh-bridge rail which is 10 inches higher than the line level, the reaction is reduced by 10% only. Find the distance of the C.G. of the loaded wagon from the line level. Take the wheel diameter as 4 ft. and add 1.5 inches with the gauge to get the mean distance between the wheels in a pair. *Ans.* 3 ft. 2.4 ins.

321. An almirah with its contents weighs 250 lbs. and has an over-all dimension—$2\frac{1}{2}$ ft. $\times$ $1\frac{1}{2}$ ft. $\times$ 6 ft. Its four legs touch the floor level just below the four corners of the almirah. Two of its legs $1\frac{1}{2}$ ft. apart are placed on the floor of a weighing machine 6 inches high. The machine indicates a load of 75 lbs. on it. Find the height of the C.G. of the body from the ground. If four men of equal height hold the almirah with its four corners in its lying position having the back side down, on their shoulders and carry it from the ground floor to the first floor where the slope of the staircase is 30°, find the load on the front and rear pairs of bearers respectively. Legs are 3″ high. *Ans.* 2 ft. 8.2 ins., Upper pair—102 lbs.

Lower pair—148 lbs.

322. A motor lorry with its load weighs 2 tons. The distance between the front and rear axle is 10 feet. The coefficient of adhesion is .5. The C.G. of the lorry is determined to be at the middle of the axle distance at a height of 3.5 feet from the road level. Find the maximum acceleration that can be produced neglecting the frictional resistances in the bearings. The lorry has a front wheel drive. *Ans.* 6.86 ft. per sec. per sec.

323. If the lorry in the previous problem has a bearing resistance of 100 lbs. constant, find the velocity after 4 seconds. *Ans.* 17 miles per hour.

324. In the car of the Illus. Ex. No. 125 if the resistance in the axle bearing be a constant force of 75 lbs. find the acceleration produced.

$(P_f + R) = \mu R_2$, where R is the bearing resistance.

$$\therefore \; R_2 = \frac{P_f + R}{\mu} = \frac{\frac{w}{g} f + R}{\mu} . \text{ Again, } R_2 = R'' + \frac{w f H}{g l}$$

$$\therefore R'' + \frac{w f H}{g l} = \frac{w f}{\mu g} + \frac{R}{\mu} \text{ or, } f\left(\frac{w}{\mu g} - \frac{w H}{g l}\right) = R'' - \frac{R}{\mu} .$$

$$\therefore f = \frac{R'' - \frac{R}{\mu}}{\frac{w}{\mu g} - \frac{w H}{g l}}$$

Ans. 5 ft. per sec. per sec.

325. In the car of the above example if the wind resistance be such that it follows the law represented by the equation, $R = (.1 v^2 + 2)$ lbs., where v is the velocity of the wind per second, compute the maximum $H.P.$ that the engine can develop. *Ans.* 19.45

326. A car weighing 5000 lbs. runs with a velocity of 30 miles per hour. Tractive force is withdrawn and brakes are applied to stop the car within a distance of 60 feet. If the position of the C.G. of the car is 2.5 feet above the ground level and at a distance of $\frac{11}{20}$ l from the rear axle where l is the distance between the two axles, centre to centre, and equal to 10 feet, find the normal reaction on the front wheels to stop the car. Compare this reaction with the reaction in statics condition. *Ans.* 3375 lbs., 875 lbs. more.

CHAPTER X

MOMENT OF INERTIA

254. Moment of Inertia has been defined in article 140. The subject is treated there with respect to a particle. It was stated there that the moment of inertia of a particle is measured by the product of the mass of the particle and the square of its distance from the axis of rotation or the inertia axis. It is also called the *Second Moment of the Mass*. Generally, the letter I is used to represent the term 'moment of inertia'.

255. Moment of Inertia of a System of Particles about an Axis. If the weights of different particles be w_1, w_2, w_3, etc., and their distances from the inertia axis be r_1, r_2, r_3, etc., respectively, then, the moment of inertia of the system is equal to the sum of the moments of inertia of the individual particles, *i.e.*,

$$I = \frac{w_1}{g} r_1^2 + \frac{w_2}{g} r_2^2 + \frac{w_3}{g} r_3^2 + \ldots\ldots\ldots$$

$$= \Sigma \frac{w}{g} r^2.$$

The particles may occupy any position in space. The 'distance' means the perpendicular distance.

256. Moment of Inertia of a Rigid Body. A rigid body is composed of a number of particles. Therefore, the moment of inertia of a rigid body is the sum of the moments of inertia of all the constituent particles about the inertia axis, *i.e.*, $I = \Sigma \frac{w}{g} r^2$.

257. Moment of Inertia of a Homogeneous Rigid Body. The weight of each particle in a homogeneous body being the same, the moment of inertia can be represented in the form, $I = \int dM . r^2$, where dM represents the mass of an elementary particle. That is, $I = \frac{w}{g} \int dV . r^2$, where w is the weight per unit volume of the mass and dV is the volume of the elementary particle.

258. Radius of Gyration. If $\int dM.\, r^2 = M.\, k^2$, where M is the total mass and k is a distance, k is called the *radius of gyration* of the mass with respect to the given inertia axis. Thus, radius of gyration may be defined as the distance of a point from the inertia axis, and the total mass, if conceived to be concentrated there, will have the same moment of inertia as that of the total mass occupying a volume in space. It is generally represented by the letter k.

259. Moment of Inertia of a Homogeneous Mass with respect to a Point. If the body is referred to three rectangular co-ordinate planes, X, Y, Z, the expressions, I_x, I_y and I_z are used to denote the second moments of the mass about the three planes respectively. The subscripts denote the respective planes.

Thus, $I_x = \int dM.\, x^2$,

$I_y = \int dM\; y^2$

and $I_z = \int dM.\, z^2$, where x, y and z in the integrals denote the distances from the respective planes.

In Fig. 245 let the body be referred to the three planes passing through the given point O. Take an elementary volume of mass dM at a distance r from the point O.

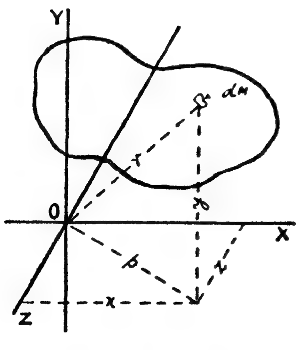


FIG. 245

Then, $r^2 = p^2 + y^2$,

but $p^2 = x^2 + z^2$,

therefore, $r^2 = x^2 + y^2 + z^2$,

$$\begin{aligned} \text{and } I_0 &= \int dM.\, r^2 \\ &= \int dM\,(x^2 + y^2 + z^2) \\ &= \int dM.\, x^2 + \int dM.\, y^2 + \int dM.\, z^2 \\ &= I_x + I_y + I_z, \end{aligned}$$

where I_0 is the moment of inertia of the mass about the point O.

Hence, $k_O{}^2 = k_x{}^2 + k_y{}^2 + k_z{}^2$, where k_O is the radius of gyration of the mass about O and k_x, k_y and k_z are the radii with respect to the three planes respectively.

Thus, the moment of inertia of a mass about any point is equal to the sum of the second moments of the mass about three rectangular planes passing through that point.

260. Moment of Inertia of a Homogeneous Mass with respect to an Axis. If the intersecting line of X and Y planes, *i.e.*, the Z-axis be the inertia axis (Fig. 246), then,

$I_z = \int dM.\, r^2$, where r is the perpendicular distance of dM from the Z-axis.

$$= \int dM\,(x^2 + y^2),$$

where x and y are the perpendicular distances of dM from X and Y planes respectively.

$$= \int dM.\, x^2 + \int dM.\, y^2$$

$$= I_x + I_y$$

and, $k_z^2 = k_x^2 + k_y^2$.

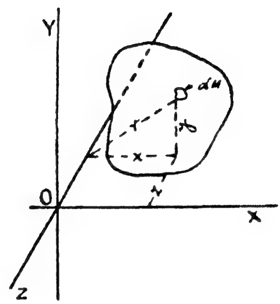


FIG. 246

Subscript z represents the axis, whereas, subscripts x and y represent the planes.

Thus, the moment of inertia of a mass about an axis is equal to the sum of the second moments of the mass about two mutually perpendicular planes whose line of intersection is the given axis.

261. Moment of Inertia of a Lamina, about various axes passing through a point in the lamina.

Take any point O in the lamina (Fig. 247). Through O draw three rectangular co-ordinate axes (X, Y, Z) such that two of them (X and Y) are on the plane of the lamina and Z is at right angles to the plane. Take an elementary mass, dM, at perpendicular distances of x and y from the X and Y planes respectively. Because the thickness of a lamina is a differential one the distances may be taken as the distances from the Y and X axes respectively. Then, the moment of inertia of the lamina about X-axis, $I_x = \int dM.\, y^2$, and that about Y-axis,

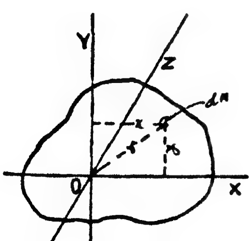


FIG. 247

$I_y = \int dM.x^2$. Hence, $I_x + I_y = \int dM(x^2 + y^2)$. But, $x^2 + y^2 = r^2$, where r is the distance of the elementary mass from the point O, *i.e.*, the Z-axis. Therefore, $I_x + I_y = \int dM.r^2 = I_z$, the moment of inertia about the Z-axis. Thus, the moment of inertia of a lamina about an axis at right angles to its planes is equal to the sum of the moments of inertia about two mutually perpendicular axes in its plane, both of which cut the previous one at the same point.

If k_x, k_y and k_z be the radii of gyration of the lamina about the three axes respectively, $k_z^2 = k_x^2 + k_y^2$.

262. Moment of Inertia of an Area. If the thickness and density be neglected and dA be substituted in place of dM, where dA is the surface area of the elementary mass, dM, then, the equation obtained in the previous article to get the value of moment of inertia will take the forms,

$$I_x = \int dA.y^2,$$

$$I_y = \int dA.x^2$$

and $$I_z = \int dA.r^2,$$

from which $$k_x^2 = \frac{\int dA.y^2}{A}$$

$$k_y^2 = \frac{\int dA.x^2}{A}$$

and $$k_z^2 = \frac{\int dA.r^2}{A}$$

also, $$I_z = I_x + I_y \qquad \text{Eq. 166}$$

Though we know that without mass there can be no moment of inertia because it is the property of mass, yet from the pictures of the equations which are just similar to those of the equations to get the values of moments of inertia and radii of gyration of mass, the forms $\int dA.x^2$, $\int dA.y^2$ and $\int dA.r^2$ which appear in computations are called the moments of inertia of an *Area* about the three axes respectively and the corresponding radii of gyration are called the radii of gyration of the area.

It is to be marked that the first moment of mass has no physical significance, but it is required in finding out the C.G. of a mass.

The second moment of mass actually signifies some physical property of mass. Again, the second moment of an area has no physical significance, but it is often found to occur in computations and with its help many important problems are solved.

263. Moment of Inertia of a Mass about Parallel Axes.

Let the two rectangular co-ordinate axes, X and Y, be drawn as shown in the diagram (Fig. 248)—X-axis passing through G, the centre of gravity of the mass. Through G draw an axis parallel to Y-axis. Let the distance between the two parallel axes be d. Take an

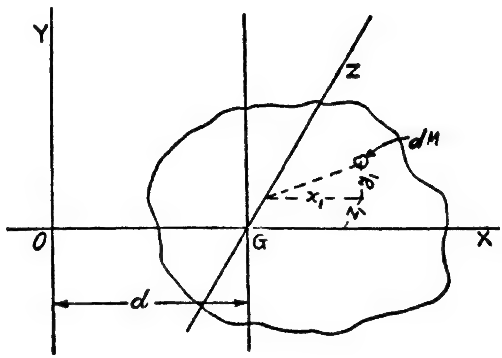


FIG. 248

elementary mass dM at distances of x_1 and z_1 from planes through G and parallel to X-plane and from Z-plane respectively. Let I_y and I_G represent the moments of inertia of the mass about the two parallel axes through O and G respectively. Then,

$$I_G = I_x + I_z \qquad \text{(Art. 260)}$$

$$= \int dM.\, x_1^2 + \int dM.\, z_1^2$$

Again, $I_y = \int dM\,(x_1 + d)^2 + \int dM.\, z_1^2$

$$= \int dM.\, x_1^2 + \int dM.\, z_1^2 + \int dM.\, d^2 + 2d \int dM.\, x_1$$

The moment about an axis through C.G. of a mass being zero, the fourth term of the equation is cancelled.

Therefore, $I_y = I_x + I_z + M\,d^2 = I_G + M\,d^2$

and $k_y^2 = k_G^2 + d^2$ ....*Eq.* 167

That is, the moment of inertia of a mass about any axis is equal to the moment of inertia of the mass about an axis passing through the C.G. of the mass and parallel to the given axis *plus* product of the mass and the distance between the two axes square.

In case of a lamina too the same relation will hold good, because the above proof is for a general case.

264. Moment of Inertia of area about Parallel Axes.

(*a*) In case of a lamina, as the thickness is very small, the consideration of the quantity I_z can be neglected. The form of the equation remains the same as, $I_y = I_G + M\,d^2$ (Fig. 249). If

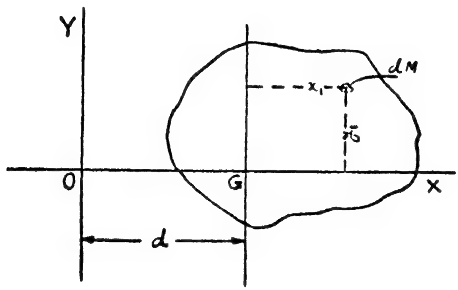


FIG. 249

now the thickness and density be not taken into account, the lamina becomes a plane surface, and the above equation takes the form as,

$I_y = I_G + A\,d^2$, where I_y and I_G are the moments of inertia of the area A about the two parallel axes respectively, and

$$k_y^2 = k_G^2 + d^2 \qquad \cdots\cdots\cdots\cdots Eq.\ 168$$

(*b*) In case (*a*) it is to be marked that the two axes were on the same plane with the lamina surface. But, the equation is true for all cases.

Take a case where the given axis is not in the same plane with the lamina surface. In Fig. 250 the given axis *O'Y'* is at a distance of *d* from the axis through C.G. (*G*) of the lamina surface parallel

to $O'Y'$. If the surface area of the elementary mass dM be dA and if it be at a distance a from the given axis, then, the moment of

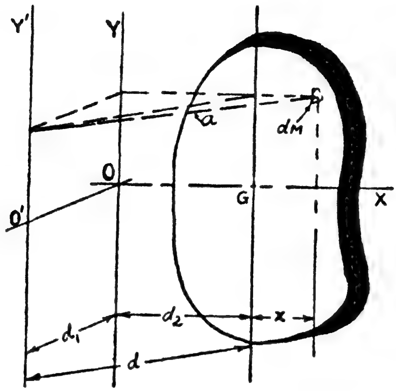


FIG. 250

inertia of this elementary area about $O'Y'$ is $dA.\,a^2$. Now, choose another parallel axis OY in the same plane with the surface of the lamina such that the common plane of $O'Y'$ and OY is at right angles to the plane surface of the lamina. Then,

$$dA.\,a^2 = dA\,\{d_1^2 + (d_2 + x)^2\}$$

Therefore, the moment of inertia of the whole surface area about $O'Y'$,

$$\begin{aligned} I_y' &= \int dA\,\{d_1^2 + (d_2 + x)^2\} \\ &= \int dA.\,d_1^2 + \int dA.\,d_2^2 + \int dA.\,x^2 + 2\,d_2 \int dA.\,x \\ &= \int dA\,(d_1^2 + d_2^2) + I_G \\ &= A\,d^2 + I_G \end{aligned}$$

And, $k^2{}_y{}' = k_G{}^2 + d^2$

265. Polar Moment of Inertia of Area. The polar moment of inertia of an area with respect to a point in its plane is the moment of inertia about an axis passing through that point and at right angles to the plane. It is evident, therefore, in article 260, I_z is the polar moment of inertia of the given area with respect to the point O, and $I_z = I_x + I_y$, subscripts denoting the three axes respectively.

266. Units of Moment of Inertia of an Area. Since the moment of inertia of an area is the product of the area and the square of the

radius of gyration, the result is expressed in the linear unit to the fourth power. *If inch or foot be used as the linear unit, the moment of inertia is expressed in* (inches)^4 — "inches to the fourth power", or (feet)^4—"feet to the fourth power". Hence, the moment of inertia expressed in (inches)^4 must be equal to the moment of inertia expressed in (feet)^4 multiplied by 12^4.

Unlike the first moment of area, the second moment, *i.e.*, the moment of inertia of an area is always positive, because it is equal to the product of the area (which is always a positive quantity) and the square of the radius of gyration. The radius of gyration may be positive or negative, but its square is always positive. It will not be out of place to mention here that for the same reason the moment of inertia of mass is always a positive quantity.

267. Deduction of Formulae for Moment of Inertia of Areas with definite shapes.

Case I. Moment of Inertia of a Rectangular Area.

(*a*) About an axis coinciding with the smaller side.

Let the bigger and the smaller sides be a and b units in length respectively (Fig. 251). Draw two rectangular co-ordinate axes, X and Y coinciding with the bigger and smaller sides respectively as shown in the diagram. Take a strip of area of width dx parallel to the Y-axis at a distance of x from it. Then, the moment of inertia of the strip about the Y-axis $= dA.\, x^2 = b.\, dx.\, x^2$.

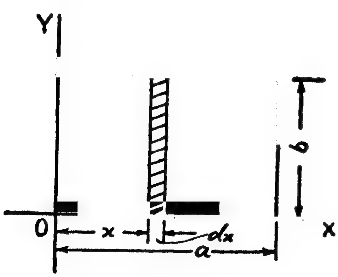


FIG. 251

Therefore, the moment of inertia of the whole area,

$$I_y = \int_0^a b.\, x^2.\, dx = \tfrac{1}{3}\, b\, a^3$$

and $k_y^2 = \dfrac{a^2}{3}$

(*b*) About an axis coinciding with the bigger side, taking the strip parallel to the X-axis, $I_x = \frac{1}{3} a\, b^3$

$$\text{and } k_x^2 = \frac{b^2}{3}$$

(*c*) About an axis passing through C.G. and parallel to the smaller arm.

$$I_y = I_G + A.\, d^2 \qquad \text{(Art. 264)}$$

$$\therefore \quad b.\, \frac{a^3}{3} = I_G + b.\, a.\, \left(\frac{a}{2}\right)^2$$

$$\text{or, } I_G = \frac{b\, a^3}{3} - \frac{b\, a^3}{4} = \frac{b\, a^3}{12}$$

$$\text{and, } k^2{}_G = \frac{a^2}{12}$$

Similarly, about an axis parallel to the bigger arm,

$$I_G = \frac{a\, b^3}{12} \quad \text{and,} \quad k^2{}_G = \frac{b^2}{12}$$

It is to be marked here that of the two values the latter one is smaller, because b is smaller than a. Again, of all the moments of inertia about axes parallel to either smaller or bigger arms, this value is the smallest.

(*d*) Polar moment of inertia with respect to C.G.

Shift the X and Y axes to pass through C.G.,

then, $I_z = I_x + I_y$ (Art. 265)

$$= \frac{b\, a^3}{12} + \frac{a\, b^3}{12} = \frac{a\, b}{12}(a^2 + b^2)$$

$$\text{and, } k_z^2 = \frac{a^2 + b^2}{12}$$

Case II. Circular area.

(*a*) Polar moment of inertia with respect to the centre.

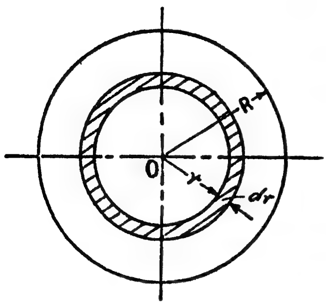


FIG. 252

Take a thin circular strip of width dr at a radius r and choose the rectangular axes X and Y to pass through the centre O and in the plane of the circular area (Fig. 252). Let the radius of the circular area be R. The area of the strip $= 2\pi r\, dr$, and its moment of inertia about an axis passing through O and at right angles to the plane is equal to $2\pi r\, dr\, r^2$. Therefore, the moment of inertia of the whole area about the same axis,

$$I_z, \text{ i.e., } I_o = \int_0^R 2\pi r\, dr\, r^2 = \frac{\pi R^4}{2} = \frac{\pi D^4}{32},$$

where D is the diameter of the area,

$$\text{and, } k_z^2 \text{ i.e., } k_o^2 = \frac{\pi R^4}{2} \div \pi R^2 = \frac{R^2}{2} = \frac{D^2}{8}$$

(a_1) Polar moment of inertia of a sector of an angle θ at the centre with respect to the same point as before.

Choosing a strip as before, the area becomes equal to $r\,\theta\, dr$.

$$\text{Therefore, } I_z = \int_0^R r\,\theta\, dr\, r^2 = \frac{\theta R^4}{4} \text{ and, } k_z^2 = \frac{R^2}{2}$$

$$\text{When } \theta = 90°, \; I_z = \pi \frac{R^4}{8} \text{ and, } k_z^2 = \frac{R^2}{2}$$

$$\text{and when } \theta = 180°, \; I_z = \pi \frac{R^4}{4} \text{ and, } k_z^2 = \frac{R^2}{2}$$

(b) About a diameter.

$I_z = I_x + I_y$ (Art. 265), but I_x must be equal to I_y, because the area is symmetrical about both the axes.

$$\text{Therefore, } I_x = I_y = \tfrac{1}{2} I_z = \frac{\pi R^4}{4}$$

$$= \frac{\pi D^4}{64}$$

$$\text{and, } k_x^2 = k_y^2 = \frac{R^2}{4}$$

Case III. Triangular area.

(*a*) About the base.

Let the X-axis coincide with the base of the triangle (Fig. 253). Take a strip of width dy and length b_1 at a distance y from the X-axis and parallel to it. Let the length of the base be b and the altitude be h. Then, $\frac{b_1}{b} = \frac{h-y}{h}$

$$\therefore \quad b_1 = b \cdot \frac{h-y}{h}$$

The area of the strip

$$= b_1 . dy = \frac{b}{h}(h-y)\,dy$$

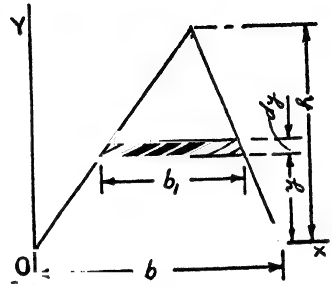


FIG. 253

$$\therefore \quad I_x = \int_0^h \frac{b}{h}(h-y)\,y^2\,dy = \frac{bh^3}{12},$$

$$\text{and,} \qquad k_x^2 = \frac{bh^3}{12} \div \frac{bh}{2} = \frac{h^2}{6}.$$

(*b*) About an axis through C.G. and parallel to the base.

$I_x = I_G + A\,d^2$, where I_G is the moment of inertia about an axis through the C.G., parallel to the base, and A is the area of the triangle.

$$\therefore \quad I_G = I_x - A\,d^2 = \frac{bh^3}{12} - \frac{bh}{2} \times \tfrac{1}{9}h^2 = \frac{bh^3}{36},$$

$$\text{and,} \quad k_G^2 = \frac{bh^3}{36} \div \frac{bh}{2} = \frac{h^2}{18}.$$

Case IV. Elliptical area.

(*a*) About the minor axis.

Let the X and Y axes coincide with the major and minor axes respectively (Fig. 254). Also let half of the lengths of the major and minor axes be equal to a and b respectively. The equation of

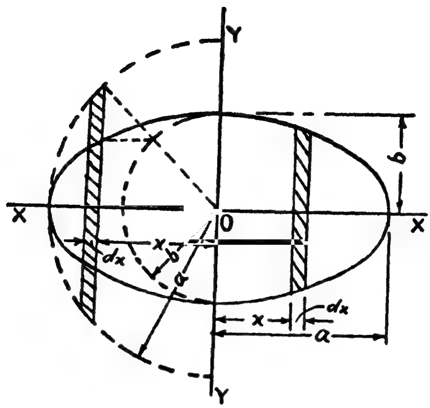


FIG. 254

an ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

Therefore, $y^2 = b^2 - \dfrac{b^2x^2}{a^2}$

or, $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$

Take a strip of area parallel to the minor axis and at a distance x whose width is equal to dx (Fig. 254, right-hand portion) and let its length be equal to y'. Thus,

the area of the strip $= 2\dfrac{b}{a}\sqrt{a^2 - x^2}.\,dx$, as, $y' = 2\dfrac{b}{a}\sqrt{a^2 - x^2}.$

Therefore, $I_y = \dfrac{2b}{a}\displaystyle\int_{-a}^{+a} x^2\sqrt{a^2 - x^2}.\,dx$

Let $x = a\sin\theta$, $\therefore$ $dx = a\cos\theta\,d\theta$

$$\therefore \quad I_y = \frac{2b}{a}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} a^2\sin^2\theta\,\sqrt{a^2 - a^2\sin^2\theta}.\;a\cos\theta\,d\theta$$

$$= \frac{2b}{a}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} a^4\sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{2b}{a} \cdot a^4 \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \tfrac{1}{4} (2 \sin \theta \cos \theta)^2 \, d\theta$$

$$= \tfrac{1}{2} ba^3 \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin^2 2\theta \, d\theta$$

$$= \frac{ba^3}{2} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \tfrac{1}{2} . \, 2 \sin^2 2\theta \, d\theta$$

$$= \frac{ba^3}{4} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} (1 - \cos 4\theta) \, d\theta$$

$$= \frac{ba^3}{4} \left| \theta - \frac{\sin 4\theta}{4} \right|_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}$$

$$= \frac{ba^3}{4} \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\pi ba^3}{4}$$

and, $k_y^2 = \dfrac{\pi ba^3}{4} \div \dfrac{\pi}{4} ab = \dfrac{a^2}{4}$.

(*b*) About major axis (*X*-axis).

Proceeding in the same way as before, $I_x = \dfrac{\pi ab^3}{4}$ and $k_x^2 = \dfrac{b^2}{4}$.

(*c*) Polar moment of inertia with respect to the C.G.

$$I_z = I_x + I_y = \frac{\pi a b^3}{4} + \frac{\pi a^3 b}{4} = \frac{\pi a b}{4} (a^2 + b^2)$$

and, $k_z^2 = \dfrac{a^2 + b^2}{4}$.

(*d*) Moments of inertia in terms of major and minor axes.

Let h_1 and h_2 be the major and minor axes respectively, then,

$$a = \frac{h_1}{2} \text{ and } b = \frac{h_2}{2}$$

1. $I_y = \frac{\pi a^3 b}{4} = \frac{\pi h_1^3 h_2}{4 \times 8 \times 2} = \frac{\pi}{64} b_1^3 b_2$ and $k_y^2 = \frac{h_1^2}{16}$

2. $I_x = \frac{\pi}{64} b_2^3 b_1$ and $k_x^2 = \frac{h_2^2}{16}$

3. $I_z = \frac{\pi}{64} b_1 b_2 (b_1^2 + b_2^2)$ and $k_z^2 = \frac{h_1^2 + h_2^2}{16}$.

Alternative Method

On the left-hand side of the Y-axis (Fig. 254) the graphical method of constructing an ellipse is shown. Take a strip of area at a distance x from the Y-axis and parallel to it, whose length is $2H$ and width dx. Let the length of the portion of the strip within the elliptical area be $2h$.

Then, from the similarity of triangles, $\frac{b}{a} = \frac{h}{H}$

Now, $\frac{\text{The moment of inertia of the strip within the ellipse}}{\text{The moment of inertia of the whole strip}}$

about X-axis, is equal to $\dfrac{\frac{1}{12}\, dx.\ 8.\ h^3}{\frac{1}{12}\, dx.\ 8.\ H^3} = \frac{h^3}{H^3} = \frac{b^3}{a^3}$

Therefore, the moment of inertia of the ellipse about XX (the major axis), $I_x = \frac{b^3}{a^3} \times$ moment of inertia of the circular area of radius a about the diameter along XX.

$$\therefore\ I_x = \frac{b^3}{a^3} \times \frac{\pi a^4}{4} = \frac{\pi}{4} a b^3$$

Similarly, the moment of inertia about YY (the minor axis),

$$I_y = \frac{\pi}{4} b a^3$$

268. Moment of Inertia of a Composite Area. If a composite area is made up of a number of simpler areas, such as triangle, rectangle, etc. (approximate rolled sections are of that kind), then, the

moment of inertia of the whole area is equal to the sum of the moments of inertia of the individual areas. The method of determining the moment of inertia in those cases is explained by an illustrated example below.

Illus. Ex. 126. *Determine the moment of inertia of the area in Fig. 234 about an axis coinciding with the base horizontal line and also about an axis parallel to it passing through the C.G. of the area, and thirdly, about an axis passing through C.G. and at right angles to the previous one.*

The area is divided into three rectangular areas—top horizontal, middle vertical and the base horizontal.

1. TOP AREA—

(*i*) Moment of inertia about an axis parallel to the inertia axis and passing through C.G.

$$\frac{1}{12} \times 4 \times \left(\tfrac{3}{4}\right)^3 = \frac{9}{64} \text{ (in)}^4 \qquad (\textit{Case I-c})$$

(*ii*) About the inertia axis—(*parallel axes*)

$$\frac{9}{64} + \left(4 \times \frac{3}{4}\right) \times \left(1 + 6 + \frac{3}{8}\right)^2$$

$$= \frac{9}{64} + \frac{10443}{64} = \frac{2613}{16} = 163.3 \text{ (in)}^4$$

2. MIDDLE AREA—

(*i*) About an axis passing through C.G. and parallel to inertia axis,

$$= \tfrac{1}{12} \times \tfrac{3}{4} \times 6^3 = 13.5 \text{ (in)}^4$$

(*ii*) About the inertia axis,

$$= 13.5 + \left(6 \times \tfrac{3}{4}\right) \times (1 + 3)^2 = 85.5 \text{ (in)}^4$$

3. BASE AREA—about the inertia axis,

$$= \tfrac{1}{3} \times 6 \times 1 = 2 \text{ (in)}^4 \qquad (\textit{Case I-a})$$

Therefore, the total moment of inertia about the axis coinciding with the base horizontal line, $I_a = 163.3 + 85.5 + 2$

$$= 250.8 \text{ (in)}^4.$$

Alternative Method—(By integration)

$$I_a = \int_7^{7\frac{3}{4}} 4 . y^2 . dy + \int_1^7 \tfrac{3}{4}\, y^2 . dy + \int_0^1 6\, y^2 . dy$$

$$= \tfrac{4}{3}\left\{\left(\frac{31}{4}\right)^3 - 7^3\right\} + \tfrac{1}{4}(7^3 - 1) + 2$$

$$= \frac{2613}{16} + \frac{342}{4} + 2 = 163.3 + 85.5 + 2 = 250.8 \ (\text{in})^4$$

II. Moment of inertia of the area about an axis passing through C.G. of the area and parallel to the base line.

$\bar{y} = 3.194$ inches

$$I_{GX} = 250.8 - (4 \times \tfrac{3}{4} + 6 \times \tfrac{3}{4} + 6 \times 1) \times (3.194)^2 \ (\textit{parallel axes})$$

$$= 250.8 - 137.6 = 113.2 \ (\text{in})^4.$$

III. Moment of inertia about an axis passing through C.G. and at right angles to the previous one, *i.e.*, *Y*-axis.

$$I_{GY} = \frac{1}{12}\left\{\tfrac{3}{4} \times 4^3 + 6 \times (\tfrac{3}{4})^3 + \tfrac{3}{4} \times 6^3\right\} \qquad (\textit{Case I-c})$$

$$= \frac{1}{12}(48 + 2.53 + 162) = \frac{1}{12} \times 212.53 = 17.71 \ (\text{in})^4.$$

Illus. Ex. 127. *Find the moment of inertia of the areas in Fig. 255 with respect to a horizontal axis passing through the C.G.*

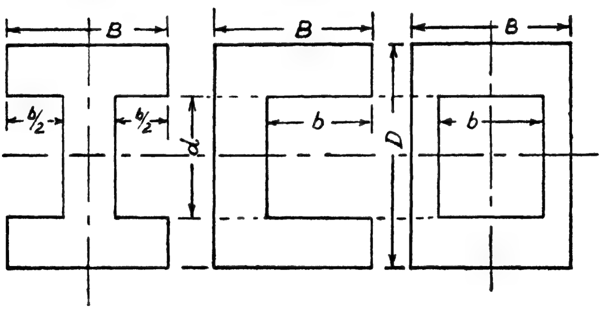


FIG. 255

$$I_{GX} = 2\left\{\int_{\frac{d}{2}}^{\frac{D}{2}} B y^2\,dy + \int_0^{\frac{d}{2}} (B - b) y^2\,dy\right\}$$

$$= 2\left\{B\left|\frac{y^3}{3}\right|_{\frac{d}{2}}^{\frac{D}{2}} + (B - b)\left|\frac{y^3}{3}\right|_0^{\frac{d}{2}}\right\}$$

$$= \frac{2}{3}\left\{\frac{BD^3}{8} - \frac{Bd^3}{8} + \frac{Bd^3}{8} - \frac{bd^3}{8}\right\}$$

$$= \frac{1}{12}\,(BD^3 - bd^3).$$

Illus. Ex. 128. *Prove that the radius of gyration of an area enclosed by two concentric circles of radii r_1 and r_2 with respect to an axis passing through the common centre of the circles and at right angles to the area plane, is equal to the mean radius when r_1 and r_2 are very nearly equal to each other.*

Take, $r_1 > r_2$

The radius of gyration square of the area about the axis

$$= \frac{\pi}{2}\,(r_1^4 - r_2^4) \div \pi\,(r_1^2 - r_2^2)$$

$$= \frac{r_1^2 + r_2^2}{2} = \left(\frac{r_1 + r_2}{2}\right)^2 + \left(\frac{r_1 - r_2}{2}\right)^2$$

Therefore, when r_2 is very nearly equal to r_1, the radius of gyration becomes approximately equal to $\frac{1}{2}(r_1 + r_2)$, which is the mean radius of the area.

In case of thin cylindrical bodies (thickness is negligible in comparison with the diameter), for all practical purposes k is taken as r_1, the external diameter of the cylinder, *i.e.*, of the transverse cross-sectional area.

269. Deduction of Formulae for Moment of Inertia of Bodies with regular shapes.

Case I. Thin Uniform Straight Rod.

About an axis passing through one end.

Let the rectangular co-ordinate axes be chosen in such a way that they lie on the same plane with the centre line of the rod. Also let the centre line make an angle θ with the Y-axis (inertia axis)—Fig. 256.

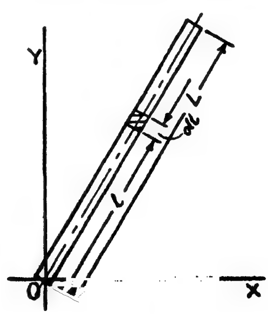


FIG. 256

The origin of the axes is at the end of the centre line, O. By the term 'thin rod' it is understood that the mass of the rod is assumed to be concentrated along the centre line of the rod. Now, if w be the weight of the rod per unit length, the total weight of the rod, $W = w.L$, where L is the total length of the rod.

Take an elementary length, dl at a distance l from the origin, O. Then,

the weight of this elementary length is $w.\,dl$. Moment of inertia of this elementary portion about the inertia axis $= l^2 \sin^2\theta.\ \dfrac{w}{g}\,dl$.

$$\text{Therefore, } I_y = \int_0^L l^2 \sin^2\theta.\ \frac{w}{g}\,dl = \tfrac{1}{3}\,.\ \frac{w\,L^3}{g}\ \sin^2\theta$$

$$= \tfrac{1}{3}\,.\ \frac{W\,L^2}{g}\ \sin^2\theta = \tfrac{1}{3}\,M\,L^2 \sin^2\theta$$

$$\text{and } k_y^2 = \tfrac{1}{3}\ L^2 \sin^2\theta$$

(*a*) When the inertia axis is perpendicular to the centre line, *i.e.*, when $\theta = \dfrac{\pi}{2}$, $I_y = \tfrac{1}{3}\dfrac{W}{g}\ L^2$.

When $\theta = 0$, moment of inertia becomes zero, which is an absurd thing. This can only be explained in the way that because the rod is thin, the cross-sectional dimension is nil, *i.e.*, it becomes a straight line and coincides with the axis of rotation and the determination of radius of gyration becomes an impossibility. If there is a positive value for the cross-sectional dimension, the rod becomes a cylindrical body and the solution is shown in the next problem.

(*b*) If the inertia axis passes through the centre of the rod (*i.e.*, C.G.) and if $\theta = \dfrac{\pi}{2}$, following the principle of the parallel axes,

$$I_y = I_G + \frac{W}{g}\,.\ \frac{L^2}{4}\,.$$

$$\therefore\ I_G = \frac{W\,L^2}{3\,g} - \frac{WL^2}{4g}$$

$$= \frac{1}{12}\,.\ \frac{W}{g}\ L^2$$

Case II. Cylindrical Body.

(*a*) About the axis of the body.

Let the X-axis coincide with the axis of the cylinder and the

Y-axis lie on the plane surface of the body (Fig. 257). Take a thin slice of the body at right angles to the axis at a distance x from the origin, O and let its width be dx. Then, if w be the weight of the body per unit volume, the weight of the thin slice is $\pi . R^2 . dx . w$, where R is the radius of the cylinder. Hence, the moment of inertia of the slice (polar moment) about the X-axis is equal to

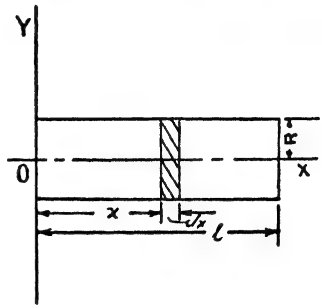


FIG. 257

$$\frac{w}{g} \pi R^2 . dx . \frac{R^2}{2}$$

Therefore, $I_x = \int_0^l \frac{w}{g} \pi R^2 . \frac{R^2}{2} dx$, where l is the length of the body.

$$= \frac{w}{g} \pi R^2 l . \frac{R^2}{2}$$

$$= \frac{W}{g} . \frac{R^2}{2}$$

$$\therefore \quad k_x^2 = \frac{R^2}{2}$$

Alternative Method

(a) Take the slice at a distance x from the Y-axis and let its thickness be dx. Then the moment of inertia of the slice about X-axis can be represented by, $\int_0^R 2 \pi r . dr . r^2 . dx . \frac{w}{g}$, where r is the polar distance of a circular ring in the slice. Therefore, the moment of inertia of the whole cylinder,

$$I_x = \int_0^l \int_0^R 2 \pi r . dr . r^2 . dx . \frac{w}{g}$$

$$= 2 \pi \frac{w}{g} \int_0^l \int_0^R r^3 . dr . dx$$

$$= 2\pi \frac{w}{g} \cdot \frac{r^4}{4} \cdot l = \pi r^2 l . \frac{w}{g} \cdot \frac{r^2}{2} = \frac{W}{g} \cdot \frac{r^2}{2}$$

(*b*) About the *Y*-axis.

The moment of inertia of the slice about a diameter parallel to the *Y*-axis and in the plane surface of the slice is equal to

$$\frac{w}{g} \pi R^2 . dx . \frac{R^2}{4} = \frac{w}{g} \pi \frac{R^4}{4} . dx$$

Its moment of inertia about the *Y*-axis (according to the principle of parallel axes) $= \frac{w}{g} \cdot \frac{\pi R^4}{4} . dx + \frac{w}{g} \pi . R^2 . dx . x^2$

$$\text{Therefore, } I_y = \int_0^l \frac{w}{g} \cdot \frac{\pi R^4}{4} dx + \int_0^l \frac{w}{g} \pi R^2 . x^2 . dx$$

$$= \frac{w}{g} \cdot \frac{\pi R^4}{4} l + \frac{w}{g} . \pi R^2 . \frac{l^3}{3}$$

$$= \frac{w}{g} . \pi . R^2 l \left(\frac{R^2}{4} + \frac{l^2}{3} \right)$$

$= \frac{W}{g} \left(\frac{R^2}{4} + \frac{l^2}{3} \right)$, where W is the total weight of the solid.

$$\therefore \quad k_y^2 = \frac{R^2}{4} + \frac{l^2}{3}$$

(*c*) About an axis passing through the C.G. of the body at right angles to *X*-axis.

$$I_y = I_G + \frac{W}{g} \left(\frac{l}{2} \right)^2 \quad \text{or,}$$

$$\frac{W}{g} \left(\frac{R^2}{4} + \frac{l^2}{3} \right) = I_G + \frac{W}{g} . \frac{l^2}{4} .$$

$$\therefore \quad I_G = \frac{W}{g} \left(\frac{R^2}{4} + \frac{l^2}{3} - \frac{l^2}{4} \right) = \frac{W}{4g} \left(R^2 + \frac{l^2}{3} \right).$$

$$\therefore \quad k_G^2 = \tfrac{1}{4} \left(R^2 + \frac{l^2}{3} \right)$$

Case III. Hollow Cylindrical Bodies.

About the axis of the cylinder (Fig. 258).

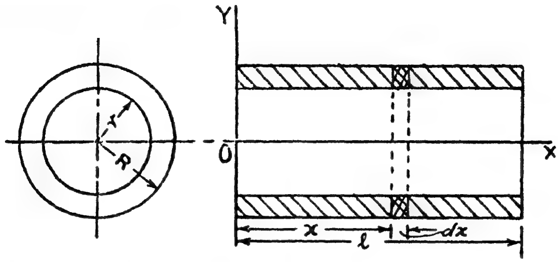


FIG. 258

The polar moment of inertia of the cross-sectional area of the cylinder (X and Y axes are taken just in the same way as was done in the previous case),

$$= \int_r^R 2\pi r_1 \, dr_1 . r_1^2,$$ where dr_1 is the thickness of a circular ring area at a radius r_1.

$$= 2\pi \int_r^R r_1^3 \, dr_1 = \frac{\pi (R^4 - r^4)}{2}$$

Therefore, the radius of gyration square $= \dfrac{\pi (R^4 - r^4)}{2} \div \pi (R^2 - r^2)$

$$= \frac{R^2 + r^2}{2}$$

Now, considering the case of the hollow cylinder in the same way with the solid cylinder,

$$I_x = \int_0^l \frac{w}{g} \pi (R^2 - r^2) \, dx . \frac{R^2 + r^2}{2}$$

$$= \frac{w}{g} \pi (R^2 - r^2) \, l . \frac{R^2 + r^2}{2} = \frac{W}{g} \frac{R^2 + r^2}{2}$$

N.B. $k_x^2 = \dfrac{R^2 + r^2}{2} = \left(\dfrac{R + r}{2}\right)^2 + \left(\dfrac{R - r}{2}\right)^2$

When the cylinder is very thin, $\left(\dfrac{R - r}{2}\right)^2$ is practically equal to

zero, and, therefore, k_x^2 is approximately equal to $\left(\frac{R+r}{2}\right)^2$.

Thus, with the thinness of the wall k_x approaches the value, $\frac{R+r}{2}$, *i.e*, the mean radius.

Case IV. Solid Sphere (Fig. 259).

About any diameter :—

Let the two rectangular axes pass through the centre of the sphere. Divide the sphere into thin slices of thickness dx perpendicular to the

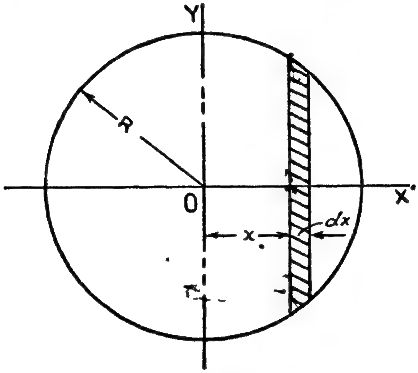


FIG. 259

X-axis. Take such a slice at a distance x from the Y-axis. The radius of the slice $= \sqrt{(R^2 - x^2)}$, where R is the radius of the sphere. The moment of inertia of this slice about the X-axis

$$= \frac{w}{g} \pi (R^2 - x^2)\, dx. \frac{R^2 - x^2}{2},$$

where w is the weight of the unit volume of the solid.

$$\text{Therefore, } I_x = \int_{-R}^{+R} \frac{w}{g} \pi (R^2 - x^2)\, dx \times \frac{R^2 - x^2}{2}$$

$$= \frac{w}{g} \cdot \frac{\pi}{2} \int_{-R}^{+R} (R^4 - 2 R^2 x^2 + x^4)\, dx$$

$$= \frac{8}{15} \frac{w}{g} \pi R^5 = \frac{w}{g} \cdot \frac{4}{3} \pi R^3 \times \frac{2}{5} R^2$$

$$= \frac{2}{5}\frac{W}{g} R^2, \quad \text{and} \quad k_x^{\ 2} = \frac{2}{5} R^2$$

Case V. Thin hollow spherical body.

Suppose the spherical body in the previous case to be hollow, the thickness being t. Then, other conditions remaining the same, the polar moment of inertia of the slice $= 2\,\pi\,R.\,dx.\,t.\ \frac{w}{g}\ (R^2 - x^2)$.

Therefore, the total moment of inertia of the whole body,

$$I_x = \int_{-R}^{+R} 2\,\pi\,R\,(R^2 - x^2)\,t.\ \frac{w}{g}\,.dx$$

$$= 2\,\pi\,R\,t.\,\frac{w}{g}\int_{-R}^{+R} (R^2 - x^2)\,dx$$

$$= 2\,\pi\,R\,t.\ \frac{w}{g}\ (2\,R^3 - \tfrac{2}{3}\,R^3) = \tfrac{8}{3}\,\pi\,R^4\,t.\ \frac{w}{g}$$

$$= 4\,\pi\,R^2\,t.\ \frac{w}{g} \times \tfrac{2}{3}\,R^2 = \tfrac{2}{3}\,\frac{W}{g}\,R^2, \quad \text{and,} \quad k_x^{\,2} = \tfrac{2}{3}\,R^2$$

Case VI. Right Circular Cone (Fig. 260).

(*a*) About its axis :—

Let the X-axis coincide with the axis of the cone. Divide the cone into thin slices at right angles to the inertia axis. The moment of inertia (polar) of a slice at a distance x from the origin, O, the vertex of the cone, whose thickness is dx is $\pi\,r^2\,\frac{x^2}{h^2}\,dx\,\frac{w}{g}\,.\,\frac{r^2\,x^2}{2h^2}$, where h is the altitude of the cone and r_1 is the radius of the slice., whence $r_1 = r\,\frac{x}{h}$

$$\therefore\ \ I_x = \int_0^h \frac{\pi}{2}\,.\,\frac{r^4}{h^4}\,\frac{w}{g}\,.\,x^4.\,dx.$$

$$= \frac{\pi}{2}\,.\,\frac{r^4}{h^4}\,.\,\frac{w}{g}\int_0^h x^4.\,dx.$$

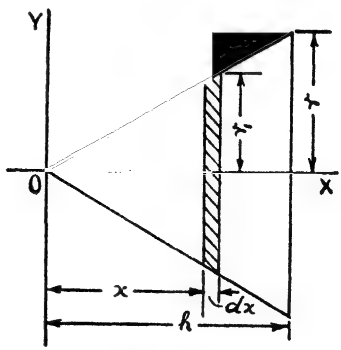


FIG. 260

$$= \frac{\pi}{2} \cdot \frac{r^4}{h^4} \cdot \frac{w}{g} \cdot \frac{h^5}{5} = \frac{\pi}{10} \cdot r^4 h \frac{w}{g}$$

$$= \frac{w}{g} \cdot \tfrac{1}{3} \pi r^2 h \times \frac{3}{10} r^2 = \frac{3}{10} \frac{W}{g} r^2$$

and $k_x^2 = \dfrac{3}{10} r^2$

(b) About an axis (Y-axis) passing through the apex and at right angles to the axis of the body.

$$I_y = \int_0^h \pi r_1^2 \, dx \times \frac{r_1^2}{4} + \int_0^h \pi r_1^2 \, dx . x^2$$

$$= \int_0^h \pi r^2 \frac{x^2}{h^2} \cdot dx \times r^2 . \frac{x^2}{h^2} \cdot \tfrac{1}{4}$$

$$+ \int_0^h \pi r^2 . \frac{x^2}{h^2} \; x^2 \, dx$$

$$= \frac{\pi r^4}{4h^4} \int_0^h x^4 \, dx + \frac{\pi r^2}{h^2} \int_0^h x^4 \, dx$$

$$= \frac{\pi r^4 h}{20} + \frac{\pi r^2 h^3}{5} = \tfrac{1}{3} \pi r^2 h \times \tfrac{3}{5} \left(\frac{r^2}{4} + h^2 \right)$$

$$= V \times \tfrac{3}{5} \left(\frac{r^2}{4} + h^2 \right).$$

This multiplied by the mass per unit volume, $\dfrac{w}{g}$, gives the moment of inertia of the body about Y-axis,

$$i.e. \quad I_y = \tfrac{3}{5} \frac{W}{g} \left(\frac{r^2}{4} + h^2 \right)$$

Illus. Ex. 129. *Fig. 236 is the full sectional view of a body made of cast iron having two distinct portions. The lower portion has the shape of a circular disc, and the upper one is hemispherical, the plane surface of which exactly coincides with the upper face of the disc. Compute the moment of inertia of the body about an axis coinciding with the diameter of the lower face of the disc. Cast iron weighs 450 lbs. per cu. feet.*

Let I_1 and I_2 be the moments of inertia of the upper and lower portions respectively. Then, if I be the moment of inertia of the whole body, $I = I_1 + I_2$.

Take a slice in the hemispherical portion of width dy at a distance of y from the plane surface of the portion and parallel to it. Then, the volume of the slice $= \pi (r^2 - y^2)\, dy$. If w be the weight per unit volume of the material, then, the moment of inertia of the upper portion with respect to an axis parallel to the given axis and on the plane surface of the hemisphere is

equal to, $\displaystyle\int_0^r \pi (r^2 - y^2)\, dy . \frac{w}{g} . \frac{r^2 - y^2}{4} + \int_0^r \pi (r^2 - y^2)\, dy . \frac{w}{g} . y^2,$

where r is the radius of the hemisphere.

$$= \frac{w}{g}\left\{ \frac{\pi}{4} \int_0^r (r^4 - 2r^2y^2 + y^4)\, dy + \pi \int_0^r (r^2y^2 - y^4)\, dy \right\}$$

Now, integrating the moment of inertia of the portion with respect to the axis referred to,

$$= \frac{w}{g}\left\{ \frac{\pi}{4} . \frac{8}{15} . r^5 + \pi\left(\frac{r^5}{3} - \frac{r^5}{5} \right) \right\} = \frac{w}{g} . \frac{4}{15} \pi r^5$$

$= \left(\frac{w}{g} . \frac{2}{3} \pi r^3 \right) . \frac{2}{5} r^2 = \frac{2}{5} . \frac{W_1}{g} r^2$, where W_1 is the weight of the hemispherical portion.

By the principle of parallel axis, the moment of inertia of the portion about an axis parallel to this axis and passing through the C G. of the hemisphere is.

$$\frac{2}{5} . \frac{W_1}{g} . r^2 - \frac{W_1}{g}\left(\frac{3}{8} r \right)^2 = \frac{93}{320} \times \frac{W_1}{g} r^2$$

Hence, by the same principle,

$I_1 = \frac{93}{320} \times \frac{W_1}{g} r^2 + \frac{W_1}{g} (h + \frac{3}{8} r)^2$, where h is the thickness of the disc.

Considering the disc as the case of a cylinder,

$I_2 = \frac{W_2}{g}\left(\frac{r^2}{4} + \frac{h^2}{3} \right)$ where W_2 is the weight of the disc

$$\text{Therefore, } I = I_1 + I_2 = \frac{W_1}{g}\left\{ \frac{93}{320} r^2 + (h + \tfrac{3}{8} r)^2 \right\} + \frac{W_2}{g}\left(\frac{r^2}{4} - \frac{h^2}{3} \right)$$

But, $W_1 = \frac{2}{3} \pi . \frac{1}{27} . 450 = 34.9$ lbs.

and $W_2 = \pi . \frac{1}{9} . \frac{1}{6} . 450 = 26.1$ lbs.

Substituting the numerical values for the notations,

$$I = \frac{34.9}{32.2}\left\{\frac{93}{320}\cdot\frac{1}{9} + \left(\frac{3.5}{12}\right)^2\right\} + \frac{26.1}{32.2}\left(\frac{1}{36} + \frac{1}{3 \times 36}\right)$$

$$= (1.083 \times .01173) + .03 = .127 + .03 = .157 \text{ unit.}$$

It is to be marked that the same result will be obtained if the volume of the body would have been considered in place of the weight,—the linear unit must be taken in feet.

APPLICATION OF MOMENT OF INERTIA

270. Kinetic Energy due to Motion of Rotation only.

In article 142 we got that the kinetic energy of a particle due to rotational motion is equal to $\frac{1}{2} I \omega^2$, where I is the moment of inertia of the particle about the axis of rotation.

In case of a system of particles weighing w_1, w_2, w_3, etc., with angular velocities of ω_1, ω_2, ω_3, etc., staying at distances of r_1, r_2, r_3, etc., from the axis of rotation respectively, the total kinetic energy of the system will be the sum of the energies of these individual particles, *i.e.*,

$$K.E._{\text{Total}} = \frac{1}{2}\left\{\frac{w_1}{g}\omega_1^2 r_1^2 + \frac{w_2}{g}\omega_2^2 r_2^2 + \frac{w_3}{g}\omega_3^2 r_3^2 + \cdots\right\}$$

$$= \frac{1}{2}\{I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 + \cdots\}$$

$$= \frac{1}{2}\Sigma I \omega^2$$

When the angular velocity is the same for all the particles,

$$K.E._{\text{Total}} = \frac{1}{2}\omega^2 \Sigma I = \frac{1}{2} I_{\text{Total}} \cdot \omega^2$$

In case of solid homogeneous body if the volume of an elementary particle be dv and if it weigh w lbs. per unit volume, then, the kinetic energy of the body when it rotates with an angular velocity of ω radians per second,

$$K.E. = \frac{1}{2}.\frac{w}{g}\omega^2 \int dV . r^2 = \frac{1}{2} V \frac{w}{g} k^2 \omega^2$$

$$= \frac{1}{2}.\frac{W}{g} k^2 \omega^2 = \frac{1}{2} I \omega^2 \qquad \cdots\cdots\cdots\cdots Eq.\ 169$$

It is to be noted that in case of linear velocity the $K.E.$ is represented by the form, $\frac{1}{2}.\frac{W}{g}v^2$, whereas in case of angular velocity it is

represented by the expression, $\frac{1}{2} . \frac{W}{g} k^2 \omega^2$. Both the forms are similar in picture and contain three portions, namely, $\frac{1}{2}$ (half), v^2 or ω^2 (velocity^2) and the remaining portion. In case of linear velocity the third portion is $\frac{W}{g}$ and in case of angular velocity it is $\frac{W}{g} k^2$. In the former case the quantity represents the mass of the body, that is, the inertia and it has been multiplied by $\frac{1}{2} v^2$, whereas in the latter case the quantity is $\frac{W}{g} k^2$ and has been multiplied by $\frac{1}{2} \omega^2$. Therefore, the quantity in the latter case might be called rotational inertia. However, we have already named it *moment of inertia* and has denoted it by I.

271. Change of K.E. with the change of rotational motion.

If the angular velocity of a body about an axis changes from ω_1 to ω_2,

$$\text{Initial } K.E. = \tfrac{1}{2}\, I\, \omega_1^2$$
$$\text{Final } K.E. = \tfrac{1}{2}\, I\, \omega_2^2$$

Therefore, the change in the $K.E. = \frac{1}{2} I\, (\omega_2^2 - \omega_1^2)$

If the change in velocity is due to the action of a constant force, P, acting tangentially to the path of rotation at a distance r from the axis and if the force acts for t seconds in which the particles of the body undergo an angular displacement, θ, then, by the principle of work, $P . r\, \theta = \frac{1}{2} I\, (\omega_2^2 - \omega_1^2)$.

$$= \tfrac{1}{2} I\, (\omega_2 + \omega_1)\, (\omega_2 - \omega_1)$$

But, $\frac{1}{2}\, (\omega_2 + \omega_1)$ is the average angular velocity and is equal to $\frac{\theta}{t}$.

Also, $(\omega_2 - \omega_1)$, which is the change of angular velocity, is equal to $\alpha . t$, where α is the angular acceleration.

Therefore, $P . r . \theta = I \times \frac{\theta}{t} \times \alpha\, t = I . \alpha . \theta$

That is, $P . r = T$ (torque) $= I\, \alpha$ ········*Eq.* 170

Thus, the moment of the effective force about the inertia axis of a rotating body is equal to the moment of inertia of the rotating body

about the axis multiplied by the angular acceleration or retardation, as the case may be, produced by the effective force.

272. Angular Momentum or Moment of Momentum of a body.

Suppose the body is rotating about the Z-axis of the three rectangular co-ordinate axes, OX, OY and OZ, with angular velocity ω (Fig. 261).

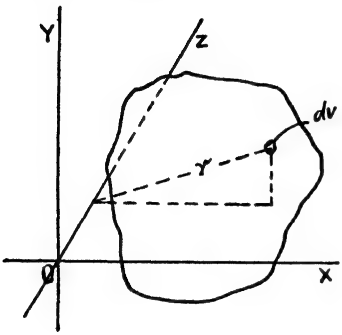


FIG. 261

Take an elementary volume dv, at a distance r from the axis of rotation. Then, the momentum of the volume is $dv. \frac{w}{g} . \omega r$, where w is the weight per unit volume of the body. Hence, the moment of this momentum $= dv. \frac{w}{g} \omega r^2$. Taking the body to be a homogeneous one, the moment of momentum of the whole body

$$= \int dv. \frac{w}{g} \omega r^2 = \omega \frac{w}{g} \int dv. r^2 = \frac{w}{g} V \omega k^2 = \frac{W}{g} k^2 . \omega$$

$$= I. \omega.$$

Angular momentum or moment of momentum of a system of bodies rotating about an axis is the sum of the angular momenta of the individual bodies and it is equal to $\Sigma I \omega$.

273. Change in Angular Momentum.

$P. r = I. \alpha = I. \frac{\omega_2 - \omega_1}{t}$, or, $P. r. t = I(\omega_2 - \omega_1)$. That is, change of angular momentum is equal to the product of torque or moment of the force and the time for which it acts. Thus, if several forces act on a body to rotate it about an axis, the sum of the torques of the individual forces multiplied by the time period for which they act gives the change in the angular momentum of the body. $P r t$, *i.e.*, $T t$ is called the *impulse of the torque.*

274. *From the foregoing discussions the three following statements may be drawn which are just similar to the expressions given in the three renowned laws of motion of Newton :*

1. It is evident that to change the angular momentum the action of a torque is required, or in other words,

> "If a body is at rest or rotates about an axis with a constant angular velocity, it will remain in its state of rest or of uniform motion until an external force is applied to produce a torque to change that state."

2. Also it has been proved that,

> "The rate of change of angular momentum is proportional to the torque applied and in the same direction with it."

3. Also it is evident that,

> "If a rotating body exerts a torque on another body then that body also will exert an equal amount of torque on the first body just in the opposite direction."

Illus. Ex. 130. *A disc fly-wheel weighing 644 lbs. has a radius of gyration of 24 inches with respect to the axis of the wheel, which is the axis of rotation of the body. Find the kinetic energy of the wheel when it is rotating at a speed of 300 r.p.m. If a brake is applied at the surface of the rim of the wheel to reduce the speed to half within a minute, determine the pressure exerted by the brake. The coefficient of friction between the brake-shoe and the rim surface is .1 and the diameter of the wheel is 4′ — 4″. What is the amount of the kinetic energy reduced?*

I. $K.E. = \frac{1}{2} \times \frac{644}{32.2} \times \left(\frac{24}{12}\right)^2 \times \left(\frac{300 \times 2\pi}{60}\right)^2$

$= 40 \times 100\,\pi^2$

$= 4000 \times 9.86 = 39440$ ft lbs.

II. $P.r.t. = I\,(\omega_2 - \omega_1)$

$= 80\,(5\pi - 10\pi)$

Therefore, $P = -\dfrac{80 \times 5\pi}{\frac{26}{12} \times 60} = -\dfrac{40\pi}{13}$. Minus sign indicates that it is acting as resistance.

Therefore, the pressure $= \dfrac{40\pi}{13 \times .1}$

$= 96.6$ lbs.

III. The kinetic energy is reduced by an amount equal to,

$40 \times \{(10\pi)^2 - (5\pi)^2\}$

$= 40 \times \pi^2\,(100 - 25)$

$= 40 \times 75 \times 9.86 = 29590$ ft. lbs.

275. Effective Force on a Rotating Body.

Let the body rotate about the Z-axis with an angular acceleration α (Fig. 262). The Y-plane is chosen in such a way that it passes through the C.G. of the body. Take an elementary mass dM in a plane parallel to XOY plane at a radial distance of r from the Z-axis. If at an instant the angular velocity of dM be ω, there are two linear

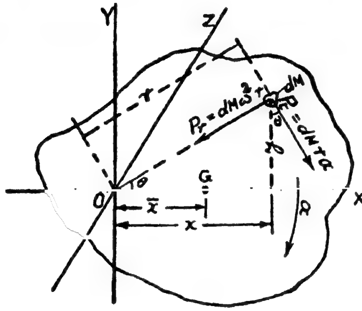


FIG. 262

accelerations in dM—one is tangential to the path of rotation by an amount, $r.\,\alpha$, and the other radial by an amount, $\omega^2\,r$, towards the axis. Therefore, the corresponding accelerating forces, P_t and P_r, are equal to $dM.\,r\,\alpha$ and $dM.\,\omega^2\,r$ respectively.

The components of these two forces along the planes parallel to X and Y planes are respectively,

$$dM.\,r.\,\alpha \cos\theta,\ \ dM.\,\omega^2\,r \sin\theta$$

and,

$$dM.\,r.\,\alpha \sin\theta,\ \ dM.\,\omega^2\,r \cos\theta$$

where, θ is the angle made by the radial line with Y-plane. Therefore, sum of the components of the effective force on the elementary masses composing the body in the horizontal direction,

$$\Sigma H = \int dM.\,r.\,\alpha \sin\theta - \int dM.\,\omega^2.\,r\cos\theta$$

But, $r\cos\theta = x$ and $r\sin\theta = y$, where x and y are the distances of dM from X and Y planes respectively. Therefore,

$$\Sigma H = \alpha \int dM.\,y - \omega^2 \int dM.\,x$$

$$= \frac{W}{g}\,\bar{y}\,\alpha - \frac{W}{g}\,\bar{x}\,\omega^2$$

Now, $\bar{y}$ being equal to zero, as X-plane passes through the C.G. of the body, $\Sigma H = -\frac{W}{g}\bar{x}\ \omega^2$ (*i.e.*, towards the axis).

Similarly, the sum of the components in the vertical direction,

$$\Sigma V = \int dM.\, r.\, \alpha \cos\theta + \int dM.\, \omega^2.\, r \sin\theta$$

$$= \alpha \int dM.\, x + \omega^2 \int dM.\, y$$

$$= \frac{W}{g}\bar{x}\ \alpha + \frac{W}{g}\bar{y}\ \omega^2$$

But, $\bar{y}$ being zero, $\Sigma V = \frac{W}{g}\bar{x}\ \alpha$

For convenience of discussion let the body be so chosen that it has got two planes of symmetry and Y and Z planes are those planes. Therefore, X and Y axes are on the Z-plane, and the line of action of the effective force must, then, be intersecting at a point on the X-axis. Then, the two components may be taken to act at any point on the line of action of the effective force, which is the resultant of the two components. ΣV is tangential to the path of rotation of that point and ΣH is radial, acting towards the axis. Now, introducing the two components at the point where the line of action of the effective force cuts the X-axis, ΣH will be found to act along the X-axis and ΣV at right angles to it. Taking moments of these two forces about the axis of rotation,

Torque, $T = \frac{W}{g}\bar{x}\ \alpha \times d$, where d is the distance of the point of application of the component forces from O.

But, $T = I\alpha$

Therefore, $\frac{W}{g}\bar{x}\ \alpha \times d = \frac{W}{g}k_o^2\,\alpha$, where k_o is the radius of gyration of the body with respect to the axis of rotation.

From which, $d = \frac{k_o^2}{\bar{x}}$.

Again, we know that, $k_o^2 = k_G^2 + x^2$

Therefore, $d = \dfrac{k_G^2 + \bar{x}^2}{x} = \bar{x} + \dfrac{k_G^2}{\bar{x}}$ ....*Eq.* 171

Illus. Ex. 131. *A thin rod, 9 feet in length, can rotate about a horizontal axis at one of its ends. It is held in a vertical position above the axis and the condition of unstable equilibrium is disturbed. The rod rotates under the influence of gravity alone. Find the tangential and radial reaction of the support when the axis of the rod makes an angle of 30° downwards with the horizontal direction. What is the linear velocity of the free end of the rod at that instant? Take the weight of the rod as 64.4 lbs.*

The moment of inertia about the axis of rotation,

$$I = \tfrac{1}{3}\,\frac{W}{g}\,l^2 = \tfrac{1}{3} \times \frac{64.4}{32.2} \times 9^2 = 54 \text{ units.}$$

Therefore, (the radius of gyration)$^2 = \frac{1}{3} l^2 = 27$

Therefore, the point of application of the effective force is at a distance $\dfrac{k^2}{\bar{x}}$, *i.e.*, $\dfrac{27}{4.5} = 6$ ft. from the axis.

Now, $T = I\,\alpha$ or, $64.4 \times 4.5 \cos 30 = 54\,\alpha$

From which, $\alpha = 4.64$ rad. per sec.

I. Hence, tangential force, $P_t = \dfrac{W}{g} \cdot x \cdot a$

$$= \frac{64.4}{32.2} \times 4.5 \times 4.647 = 41.8 \text{ lbs.}$$

Alternative Method—by the principle of moments.

$$P_t \times 6 - 64.4 \times 4.5 \cos 30 = 0$$

From which, $P_t = 41.8$ lbs.

II. By the principle of work, $W.h = \frac{1}{2} I.\omega^2$, where h is the fall of C.G

Substituting the numerical values,

$$64.4 \times 6.75 = \tfrac{1}{2} \times 54 \times \omega^2$$

From which, $\omega = 4.013$ rad. per sec.

Now, the radial component of the effective force,

$$P_r = \frac{W}{g} \cdot \bar{x} \cdot \omega^2 \qquad (Eq.\ 38)$$

Substituting the numerical values,

$$P_r = \frac{64.4}{32.2} \times 4.5 \times 16.1 = 144.9 \text{ lbs.}$$

III. The velocity of the free end of the rod at that instant,

$$v = \omega . l = 4.013 \times 9 = 36.1 \text{ feet per sec.}$$

Illus. Ex. 132. *A steel disc of weight 644 lbs. is constrained to rotate about an axis coinciding with an element of the curved surface. It is held in a position with its polar axis vertically above the axis of rotation. Next, its condition of equilibrium is disturbed and it is allowed to rotate freely due to the action of gravity alone. Determine the radial and the tangential reactions at the support when the disc rotates (1) through 120°, and (2) through 180°.* (Fig. 263).

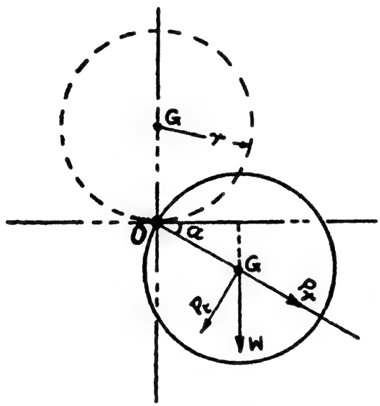


Fig. 263

$$I_o = \frac{W}{g} \cdot \frac{r^2}{2} + \frac{W}{g} r^2 = \frac{3}{2} \frac{W}{g} r^2$$

Now, the kinetic energy of the body is equal to the work done due to the gravity on the body. Thus,

$$\tfrac{1}{2} I_o \omega^2 = \tfrac{1}{2} \cdot \frac{3}{2} \frac{W}{g} r^2 \omega^2 = W (r + r \sin \theta) = W r (1 + \sin \theta)$$

$$\text{or, } \omega^2 = \frac{4}{3} \times \frac{g(1 + \sin \theta)}{r}$$

(1) The centrifugal force on the mass $= \frac{W}{g} \omega^2 r$

$$= \frac{W}{g} \times \frac{4}{3} \times \frac{g(1 + \sin \theta)}{r} \times r$$

$$= \tfrac{4}{3} W (1 + \sin \theta)$$

Therefore, the radial reaction at the support

= the centrifugal force + the component of W in the direction of the centrifugal force.

$$= \frac{4}{3} W (1 + \sin \theta) + W \sin \theta$$

$$= W \left\{\frac{4}{3}(1 + \sin \theta) + \sin \theta\right\}$$

Here, in the first case, $\theta = 30°$, therefore, the radial reaction,

$$P_r = 644 \left\{\frac{4}{3}(1 + .5) + .5\right\}$$

$$= 644 \times 2.5 = 1610 \text{ lbs.}$$

(2) Again, torque $T = I_o . \alpha$, *i.e.*, $W . r \cos \theta = \frac{3}{2} \times \frac{W}{g} r^2 . \alpha$

where α is the angular acceleration.

Therefore, $\alpha = \frac{2 g \cos \theta}{3 r}$, and the accelerating force is equal to

$\frac{W}{g} . r . \alpha = \frac{W}{g} . r . \frac{2 g \cos \theta}{3 r} = \frac{2}{3} W \cos \theta$, which is tangential to the path of rotation.

But the component of W tangential to the path of rotation is equal to $W \cos \theta$.

Thus, the accelerating force is checked by an amount,

$W \cos \theta - \frac{2}{3} W \cos \theta = \frac{1}{3} W \cos \theta$, which will produce an opposite couple about the support to reduce the acceleration.

Therefore, the tangential reaction at the support,

$$P_t = \frac{1}{3} W \cos \theta = \frac{1}{3} \times 644 \times .866 = 185.9 \text{ lbs.}$$

(3) When the disc rotates through 180°,

$$\frac{1}{2} \times \frac{3}{2} \times \frac{W}{g} r^2 . \omega^2 = W \times 2 r$$

or, $\omega^2 = \frac{8 g}{3 r}$

Therefore, $P_r = W + \frac{W}{g} \times \frac{8g}{3r} \times r = W + \frac{8}{3} W = \frac{11}{3} W$

$$= \frac{11}{3} \times 644 = 2361.3 \text{ lbs.}$$

(4) As there is no component of W tangential to the path of rotation, *i.e.*, θ being 90°, $W \cos \theta = 0$,

$$P_t = 0$$

COMPOUND PENDULUM

276. An arrangement by which the motion of a rigid body may be constrained to turn about a horizontal axis passing through a point within the mass itself, is called a *Compound Pendulum.*

If the condition of stable equilibrium of such a rigid body be disturbed by pulling it aside through some angle and allowed to move freely, then, it is found to oscillate following the law of simple harmonic motion.

277. Centre of oscillation. The centre of oscillation of a compound pendulum is the point where, if the whole mass is assumed to be concentrated, the periodic time would remain unaltered. The distance of the point from the axis of rotation is called the *length of an equivalent simple pendulum*, *i.e.*, if a simple pendulum have a length equal to this distance, it will have the same periodic time with the compound pendulum. This distance is represented by L.

278. Equivalent length and periodic time. Let A (Fig. 264) be the rigid body suspended from a horizontal pivot at O and oscillating parallel to a vertical plane and G its centre of gravity. Also let k_G and k_o be the radii of gyration of the body about G and O respectively.

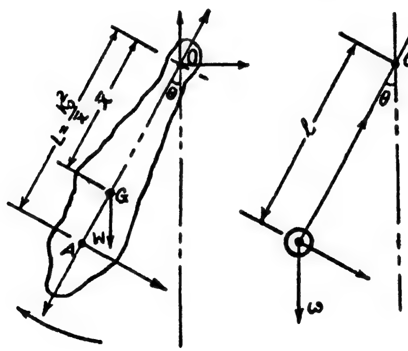


FIG. 264

If W be the weight of the body, $I_o = \dfrac{W}{g} k_o^2$ and $I_G = \dfrac{W}{g} k_G^2$.

But, $I_o = I_G + \dfrac{W}{g} x^2$. Therefore, $k_o^2 = k_G^2 + \bar{x}^2$

Let θ be the angle of deviation. In this position two forces act in the system : (1) the weight, and (2) the reaction of the pivot on the body. The sum of the moments of these forces about O is equal to $W \bar{x} \sin \theta$. But, we know that $T = I . \alpha$

The torque being in opposite direction to that of θ when increasing,

$$- W \bar{x} \sin \theta = \frac{W}{g} k_o^2 . \frac{d^2 \theta}{d t^2}$$

or, $\dfrac{d^2 \theta}{d t^2} + \dfrac{g x}{k_o^2} \sin \theta = 0$, when θ is very small, $\sin \theta = \theta$

The equation takes the form similar to that of a simple pendulum.

If L be put equal to $\dfrac{k_o^2}{\bar{x}}$, the equation becomes, $\dfrac{d^2 \theta}{d t^2} + \dfrac{g}{L} = 0$

Then by solving the equation, $\theta = \theta_a \cos \left(\sqrt{\dfrac{g . \bar{x}}{k_o^2}} . t + \varepsilon \right)$

$$= \theta_a \cos \left(\sqrt{\frac{g}{L}} \; t + \varepsilon \right).$$

where θ_a is the angular displacement for the amplitude of the swing.

Now, when the phase angle is 0, $\theta = \theta_a \cos \sqrt{\dfrac{g \bar{x}}{k_o^2}} . t$

$$= \theta_a \cos \sqrt{\frac{g}{L}} . t$$

If T be the periodic time, $T = 2 \pi \div \sqrt{\dfrac{g . \bar{x}}{k_o^2}} = 2 \pi \sqrt{\dfrac{k_o^2}{g x}}$

$$= 2 \pi \sqrt{\frac{L}{g}}$$

Again, $k_o^2 = k_G^2 + x^2$.

Therefore, $T = 2 \pi \sqrt{\dfrac{k_G^2 + \bar{x}^2}{g \bar{x}}}$ ....Eq. 173

$$L = \frac{k_o^2}{\bar{x}} = \frac{k_G^2 + \bar{x}^2}{\bar{x}} = x + \frac{k_G^2}{\bar{x}}$$

............Eq. 174

Again, in a simple pendulum the periodic time, $t = 2\pi\sqrt{\frac{l}{g}}$, where l represents the length of a simple pendulum. But, the periodic time in a compound pendulum and an equivalent simple pendulum being the same,

$$2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{L}{g}}$$

Therefore, $l = L = \bar{x} + \frac{k_G^2}{\bar{x}}$

That is, the centre of oscillation in a compound pendulum rests at a distance $\frac{k_G^2}{\bar{x}}$ beyond the position of the C.G. of the mass from the axis of rotation, along the straight line joining O and C.G.

279. *The two axes—axis of rotation and an axis parallel to it passing through the centre of oscillation—are interchangeable.*

If O be the axis of rotation (Fig. 265), G be the centre of gravity and C be the centre of oscillation, $OC = l$ and $OG = \bar{x}$

Then, $l = \frac{k_G^2}{\bar{x}} + \bar{x}$, or, $k_G^2 = \bar{x}(l - \bar{x}) \ldots\ldots (i)$

Now, if the axis of rotation be changed from O to C, and if l_1 be the distance of the centre of oscillation from this point, $\bar{x}_1$ and k_C be the distances of the centre of gravity and the radius of gyration respectively with respect to the new axis of rotation,

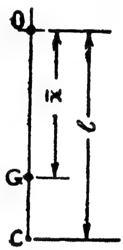


FIG. 265

$l_1 = \frac{k_C^2}{\bar{x}_1}$, but $\bar{x}_1 = (l - \bar{x})$ and $k_C^2 = k_G^2 + \bar{x}_1^{\,2}$

$$\therefore \quad l_1 = \frac{k_G^2}{\bar{x}_1} + \bar{x}_1 = \frac{k_G^2}{(l - \bar{x})} + (l - \bar{x})$$

Substituting the value of k_G^2 (*Eq. i*)

$$l_1 = \frac{\bar{x}(l - x)}{(l - \bar{x})} + (l - \bar{x}) = l.$$

Hence, the two axes are interchangeable, *i.e.*, whether O becomes the axis of rotation or C becomes the axis of rotation the periodic time remains the same.

280. Centre of Percussion. If the condition of stable equilibrium of an oscillating mass as shown in figure 266, be disturbed by a blow P, then, the body will start to oscillate in the direction shown by the arrow-head with an angular acceleration, say, α. The component of the effective force which will create acceleration is equal to $\frac{W}{g}\bar{x}\,\alpha$, and will be acting at a point which is at a distance of $\frac{k_o^2}{\bar{x}}$ from O (Art. 278) in a direction opposite to that of the motion. Now, taking the moments of the reactions at O and the force of blow P about that point, it is found that a force R,

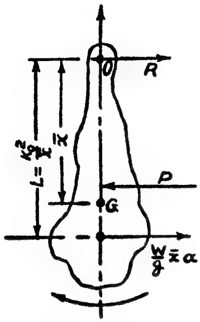


FIG. 266

which is parallel in direction to $\frac{W}{g}\bar{x}\;\alpha$, will be acting at O—to the right, if P acts above the point and to the left if P acts below. But if P acts at that point, there will be no reaction parallel to $\frac{W}{g}\bar{x}\;a$ at O. The point, where the body being struck will not produce any such reaction at the point of support, is called the *Centre of Percussion.* It is to be noted that the centre of percussion and the centre of oscillation, being at equal distance from O along the radial line, are the same point, *i.e.*, the centre of oscillation is the centre of percussion.

It is interesting to mention here that cricket players often feel the jerking while hitting balls wrongly, which is nothing but the normal reaction produced at the hand.

Illus. Ex. 133. *In a laboratory experiment to find out the moment of inertia of a fly-wheel about various axes and the corresponding radii of gyration it is given that the weight of the wheel is 322 lbs. The knife edge under the rim about which the wheel swings is at a distance of 24 inches from the axis of the wheel. If the wheel oscillates 100 times in 200 seconds, determine,* (1) *the length of an equivalent pendulum,* (2) k_o *(radius of gyration with respect to an axis coinciding with the knife edge),* (3) k_G *(radius of gyration with respect to the axis parallel to the previous one passing through the C.G. of the wheel, i.e., axis of the wheel),* (4) *moment of inertia about the knife edge, and* (5) *that about the axis of the wheel.*

I. $t =$ periodic time $= \frac{200}{100} = 2$ seconds.

$$t = 2\pi\sqrt{\frac{l}{g}}\ ,\ i.e.,\ 2 = 2\pi\sqrt{\frac{l}{g}}\ ,\ \text{or},\ \pi^2\frac{l}{g} = 1$$

Therefore, l, the equivalent length of a simple pendulum,

$$= \frac{g}{\pi^2} = 3.26 \text{ feet.}$$

II. We know that,

$$l = \frac{k_o^2}{x}\ ,\ \text{or},\ k_o^2 = l.\ x$$

$$= 3.26 \times \frac{24}{12} = 6.52 \qquad \therefore\ k_o = 2.554 \text{ ft.}$$

III. Again, $k_G^2 = k_o^2 - d^2$

$$= 6.52 - \left(\frac{24}{12}\right)^2 = 6.52 - 4$$

$$= 2.52 \qquad \therefore\ k_G = 1.586 \text{ ft.}$$

IV. If I_o be the moment of inertia about the knife edge,

$$I_o = \frac{W}{g}\ k_o^2 = \frac{322}{32\ 2} \times 6.52$$

$$= 65.2 \text{ units or slug (ft)}^2$$

V. If I_G be the moment of inertia about the axis of the wheel.

$$I_G = \frac{W}{g}\ k_G^2 = \frac{322}{32.2} \times 2.52$$

$$= 25.2 \text{ units or slug (ft)}^2$$

TORSIONAL VIBRATION

281. Vibration in a Shaft fixed at one end.

S is the shaft fixed at one end and the other end is resting on a frictionless bearing B (Fig. 267). A wheel P is rigidly fixed at the free end of the shaft. Suppose the wheel is turned to twist the shaft through an angle θ radian and then, allowed to oscillate freely. If T_C

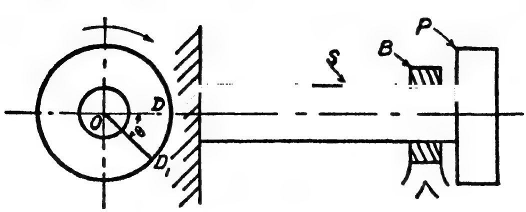


FIG. 267

be the torque required per radian of twist of the shaft, then, to twist it through θ radian the torque required $= T_C \times \theta$.

But, $T = I.\,\alpha$. Therefore, neglecting the consideration of the mass of the shaft, *i.e.*, moment of inertia of the shaft, if I_o be the moment of inertia of the wheel about the axis through O, and if W be the weight,

$$-T_C \times \theta = I_o \times \frac{d^2\,\theta}{d\,t^2}$$

That is, $\dfrac{d^2\,\theta}{dt^2} + \dfrac{T_C}{I_o}\,\theta = 0$. Now, solving the equation,

$$\theta = \theta_a \cos\left(\sqrt{\frac{T_C}{I_o}}\,.\,t + \varepsilon\right) \text{ and } t = 2\,\pi \div \sqrt{\frac{T_C}{I_o}} = 2\,\pi\sqrt{\frac{I_o}{T_C}}$$

If n be the frequency, $n = \dfrac{1}{2\pi}\sqrt{\dfrac{T_C}{I_o}}$

282. *Vibration in a shaft resting on two frictionless bearings at the two ends—wheels are rigidly fixed at the two free ends which are free to oscillate.*

Let the two wheels set in vibrations in the opposite directions with respect to the same end (Fig. 268).

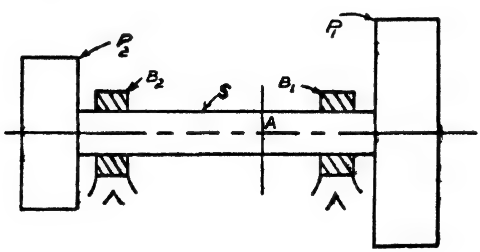


FIG. 268

Shaft S is fitted on two frictionless bearings, B_1 and B_2, and let the two wheels, P_1 and P_2, be fixed at the two free ends as shown. They are set in vibrations so that the motions are always in the opposite sense looking from the same side. Let I_1 and I_2 be the moments of inertia of P_1 and P_2 about the axis of rotation respectively. If T_1 and T_2 be the torques required per radian of twist at the two

ends respectively, then, the periodic time of the oscillation of P_1,

$$t_1 = 2\pi \sqrt{\frac{I_1}{T_1}}$$

and that of P_2, $t_2 = 2\pi \sqrt{\frac{I_2}{T_2}}$

If the two ends oscillate in the opposite sense there must be a transverse section where there is no twisting effect. That section cuts the axis of the shaft at a point. That point is called the *node*, and the plane of the section is called the *nodal plane*. Let the node be at A, which is at a distance of a from the middle of the width of the wheel P_1 and at a distance of b from that of the wheel P_2. Under the condition of motion of the system the periodic time must be equal.

Therefore, $t_1 = t_2 = 2\pi \sqrt{\frac{I_1}{T_1}} = 2\pi \sqrt{\frac{I_2}{T_2}}$

From which, $\frac{I_1}{T_1} = \frac{I_2}{T_2}$ or, $\frac{T_1}{T_2} = \frac{I_1}{I_2}$ ....*Eq.* 175

Again, it can be proved that $\frac{T_1}{T_2} = \frac{a}{b}$

(The proof is not a matter of discussion in this treatise. The proof can be obtained from any text book on Strength of Materials).

From this relation the position of A can be easily determined.

283. Damped Oscillation.

The above discussion deals with free oscillations. The consideration of the effect of the frictional resistance, which is found to be present in the mechanism of all machines, has not been considered. The effect of friction is to diminish the amplitude of the oscillation gradually, till the motion comes to a stop. This effect is known as *Damping*. Of course, with the diminution of the amplitude the periodic time decreases, but the friction in all such cases is so very small that the change in the periodic time can be neglected.

Illus. Ex. 134. *Determine the frequency of torsional vibration of a uniform straight rod, 4 lbs. in weight and 2 feet in length. The bar is suspended by a steel wire, which is fixed at the middle of the rod length, and the rod is kept balanced horizontally. The torque applied at the wire is proportional to*

the twist, and a torque of 2 lb. ft. is required to twist the wire through 114.8 degrees.

$$114.8^\circ = \frac{114.8\,\pi}{180} = 2.004 \text{ rad.}$$

Torque required to twist the wire through 1 rad.

$$= \frac{2}{2.004} = .9981 \text{ lb. ft.}$$

Now, periodic time, $t = 2\,\pi \sqrt{\frac{I}{T}}$

But $T = \frac{1}{12} \times \frac{4}{32.2} \times 2^2 = \frac{4}{96.6}$ (neglecting the mass of the wire.)

Therefore, the frequency,

$$\frac{1}{t} = \frac{1}{2\,\pi} \sqrt{\frac{96.6 \times .9981}{4}}$$

$$= .781 \text{ oscillation per second.}$$

284. Energy of a body having Plane Motion.

In Art. 55, page 74, plane motion has been defined. If the motion of a body be such that it rotates about a definite axis and at the same time the axis has a motion of translation, then, the energy of the body is calculated in the following way : Let a body rotate about the Z-axis with an angular velocity ω and also let the axis itself move in a straight direction as shown by the arrow-head with a linear velocity v_o (Fig. 269).

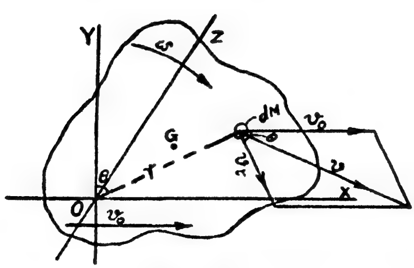


FIG. 269

Take an elementary mass, dM, at a radial distance r from O on the axis. Through O draw two other rectangular co-ordinate axes. Now, dM has two simultaneous velocities—one due to motion of rotation and the other due to motion of translation.

At an instant dM has a linear velocity tangential to the path of rotation, $v_t = \omega r$, and a linear velocity v_o in the direction of motion of translation. Their resultant velocity v is such that, $v^2 = v_o^2 + v_t^2 + 2\, v_o\, v_t \cos\theta$, where θ is the angle between the two component velocities. Then, the kinetic energy of $dM = \frac{1}{2} M v^2 = \frac{1}{2} M\ (v_o^2 + v_t^2 + 2\, v_o\, v_t \cos\theta)$.

Therefore, the kinetic energy of the whole mass,

$$K.E. = \tfrac{1}{2}\ (\int dM\, v_o^2 + \int dM\, v_t^2 + 2 \int dM\, v_o\, v_t \cos\theta)$$

$$= \tfrac{1}{2}\ (\int dM\, v_o^2 + \int dM\, \omega^2 r^2 + 2 \int dM\, v_o\, \omega\, r \cos\theta)$$

But, $r \cos\theta = y$

Therefore, $K.E. = \frac{1}{2}\ (\int dM\, v_o^2 + \int dM\, \omega^2 r^2 + 2\, v_o\, \omega \int dM\, y)$

$$= \tfrac{1}{2}\ M\, v_o^2 + \tfrac{1}{2}\, I_o\, \omega^2 + v_o\, \omega\, M\, y$$

This is the general equation for kinetic energy possessed by a body having plane motion.

It is, now, evident that if the axis of rotation passes through the C.G. of the body, $y = 0$, and the last term of the equation vanishes.

The equation becomes, $K.E. = \frac{1}{2} M\, v_o^2 + \frac{1}{2}\, I_o\, \omega^2$. ············*Eq.* 176

285. Energy of Rolling Cylindrical and Spherical bodies.

The problem is a particular case of plane motion where the axis of rotation passes through the C.G. of the body. Therefore, the kinetic energies of all such bodies are represented by the equation, $K.E. = \frac{1}{2} M\, v_o^2 + \frac{1}{2}\, I_o\, \omega^2$. But, it is always to be remembered that v_o represents the motion of translation of the axis of rotation and I_o is the moment of inertia of the body about that axis. Here, v_o represents the velocity of the axis through the C.G. of the body and I_o the moment of inertia of the body about that axis.

Alternative Method

INSTANTANEOUS AXIS AND CENTRE. When a spherical or a cylindrical body rolls over a plane surface, at any instant the body may be said to rotate about a definite axis which is called the *Instantaneous Axis* of the body at that instant. The trace of the axis on a vertical plane is called *Instantaneous Centre*. Suppose a body

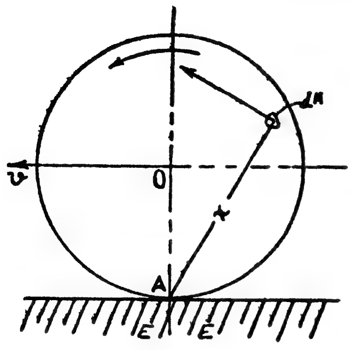


FIG. 270

(Fig. 270) of radius r rolls over the surface, EE. Let the linear velocity of the axis of the body be v. When the body is rolling in the direction shown by the arrow-head, the axis of the body through O having a linear velocity v, all the particles in the body have the same linear velocity in the same direction and in addition to that they are rotating about the axis with an angular velocity $\omega = \frac{v}{r}$.

For the instant shown the whole body may be said to rotate about the instantaneous axis through A, the point of contact, which is of course parallel to the axis through O. The angular velocity with which the particles of the body are rotating at that instant about the axis through A, is also equal to $\omega = \frac{v}{r}$

The $K.E.$ of an elementary mass, dM, of the body at a distance x from the instantaneous axis, A, is $\frac{1}{2}\, dM\, x^2\, \omega^2$.

Therefore, the $K.E.$ of the whole body $= \int \frac{1}{2}\, dM.\, x^2.\, \omega^2$

$$= \frac{1}{2}\, \omega^2 \int dM.\, x^2 = \frac{1}{2}\, \frac{W}{g}\, k^2\, \omega^2,$$ where k is the radius of gyration of the body about the axis through A.

The moment of inertia of the body about the axis through A, $I_A = \frac{W}{g} k_A^2 = I_O + \frac{W}{g} r^2$, where I_O is the moment of inertia of the body about the parallel axis through O.

$$\text{Therefore, } K.E. = \frac{1}{2}\left(I_O + \frac{W}{g} r^2\right)\omega^2 = \frac{1}{2} I_O\, \omega^2 + \frac{1}{2}\, \frac{W}{g}\, v^2$$

$$= \frac{1}{2}\, \frac{W}{g}\, (k_O^2\, \omega^2 + v^2)$$

Now, because the linear velocity of O is v, all the particles have the same velocity in the same direction and also they have the angular velocity ω which is equal to $\frac{v}{r}$. Hence, the $K.E.$ of a rolling body

is equal to the sum of the kinetic energies due to its motion of rotation about its axis and due to the motion of translation of the body.

It is to be marked that if the axis of rotation be transferred to the position of the instantaneous axis, which is represented by its trace on a vertical plane passing through the point of contact, A, the first and the last term of the general equation for kinetic energy of plane motions vanish, because v_0 denotes the velocity of the axis of rotation and it is zero in this case. Therefore, the equation takes the form—

$$K.E. = \tfrac{1}{2} I_A \omega^2$$

$$\text{But, } I_A = I_G + M r^2,$$

where I_A and I_G are the moments of inertia of the body about the axes through A and C.G. respectively.

$$\text{Hence, } K.E. = \tfrac{1}{2} (I_G + M r^2) \omega^2 = \tfrac{1}{2} I_G \omega^2 + \tfrac{1}{2} M v_0^2,$$

where v_0 is the velocity of the axis through C.G.

Thus, it is found that whether the axis of rotation passes through C.G. or it is chosen to pass through the instantaneous centre, the energy equations are the same for both the cases.

Illus. Ex. 135. *A motor car weighs 2254 lbs., of which 322 lbs. is the weight of the four wheels. Compute the kinetic energy of the car when running at a speed of 30 miles per hour. The radius of gyration of the wheels is 1 foot with respect to the axis of rotation and the maximum diameter of a wheel is 33 inches.*

$K.E.$ of the car $=$ $K.E.$ of the wheels $+$ $K.E.$ of the remaining portion.

$K.E.$ of the wheels $=$ $K.E.$ due to rotational motion $+$ $K.E.$ due to motion of translation.

Therefore, $K.E.$ of the car $=$ $K.E.$ of the total mass due to motion of translation $+$ $K.E.$ of the wheels due to motion of rotation.

$$\text{The angular velocity of the wheels, } \omega = \frac{v}{r} = \frac{44 \times 12 \times 2}{33}$$

$$= 32 \text{ radians per sec.}$$

$$\text{Hence, the required } K.E. = \tfrac{1}{2} \cdot \frac{2254}{32.2} \times 44 \times 44 + \tfrac{1}{2} \cdot \frac{322}{32.2} \times 1^2 \times 32^2$$

$$= 70 \times 22 \times 44 + 10 \times 16 \times 32$$

$$= 67760 + 5120 = 72880 \text{ ft. lbs.}$$

286. Acceleration of a Body rolling down an inclined plane.

Suppose a cylindrical or a spherical body starting from rest rolls down a plane whose inclination is θ with the level through a distance d (Fig. 271). Let the linear velocity attained by the point O, which

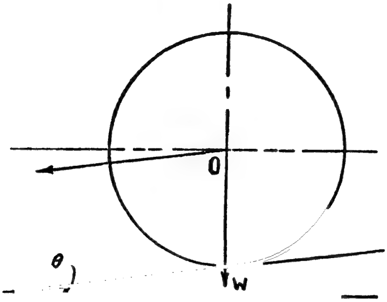


FIG. 271

represents the axis of the body or the centre as the case may be, down the plane and parallel to it be v, ω be the angular velocity of the particles about the axis and W be the weight of the body. Then,

$$K.E. = \tfrac{1}{2} \cdot \frac{W}{g} (k_o^2 \omega^2 + v^2) = \tfrac{1}{2} \cdot \frac{W}{g} \left(k_o^2 \cdot \frac{v^2}{r^2} + v^2 \right),$$

where r is the radius of the body,

$$\text{or, } K.E. = \tfrac{1}{2} \cdot \frac{W}{g} \; v^2 \left(1 + \frac{k_o^2}{r^2}\right). \qquad \cdots\cdots\cdots\cdots \textit{Eq. } 177$$

That is, the kinetic energy is the same with a body whose mass is $\frac{W}{g}\left(1 + \frac{k_o^2}{r^2}\right)$ and moves with a linear velocity v, without any rotational motion in it. This energy must be equal to the amount of work done on the body, $W \sin \theta . d$, thus,

$$W \sin \theta . d = \tfrac{1}{2} \cdot \frac{W}{g} \cdot v^2 \left(1 + \frac{k_o^2}{r^2}\right)$$

$$\therefore \quad v^2 = \frac{2\, g \sin \theta . d}{1 + \frac{k_o^2}{r^2}} = 2\, f\, d, \text{ where } f \text{ is the acceleration.}$$

$$\text{Therefore, } f = \frac{g \sin \theta}{1 + \frac{k_o^2}{r^2}}$$

$$= \frac{g\,r^2 \sin\theta}{r^2 + k_o^2} \qquad \text{............} Eq.\ 178$$

Hence, the angular acceleration,

$$\alpha = \frac{g\,r \sin\theta}{r^2 + k_o^2} \qquad \text{............} Eq.\ 179$$

Now, if P be the effective force applied tangentially to the rolling surface or applied at the axis in a direction parallel to the plane on which it rolls,

$$\text{Moment } P \times r = I\,\alpha = \frac{W}{g}\ k_o^2\,\alpha$$

$$\text{or, } P = \frac{W}{g\,r}\ k_o^2\,\alpha = \frac{W}{g\,r}\ k_o^2 . \frac{g\,r \sin\theta}{r^2 + k_o^2}$$

$$= \frac{W\,k_o^2 \sin\theta}{r^2 + k_o^2} = \frac{W \sin\theta}{1 + \dfrac{r^2}{k_o^2}}$$

$$\text{............} Eq.\ 180$$

Illus. Ex. 136. *A solid cylindrical body of diameter 2 feet and weighing 8050 lbs. rolls down a plane inclined 15° with the horizontal plane. Determine,*

(*1*) *The minimum value for the coefficient of friction so that the body can roll without slipping.*

(*2*) *The angular and the linear accelerations of the body.*

(*3*) *The kinetic energy of the body when it has rolled down through 50 feet.*

If F be the force of friction, $W \sin\theta - F = \frac{W}{g} f$ ... ... (i)

If N be the normal reaction, $N - W \cos\theta = 0$ ... ... (ii)

Taking moments about the centre, $F.r = \frac{W}{g} . \frac{r^2}{2} . \alpha$ ... (iii)

But, $f = r.\alpha$ ... ... (iv)

and $F = \mu . N$ ... ... (v)

Substituting the values of F and N, we get from the Eqs. (i) and (ii)

$$\mu = \tan\theta - \frac{f}{g \cos\theta} \qquad \text{... ...} \quad \text{(vi)}$$

Again, from the Eqs. (i) and (iii),

$$F = \frac{1}{2}\frac{W}{g} r\alpha = W\sin\theta - \frac{W}{g} f$$

From which, $f = \frac{2}{3} g \sin\alpha$ ... ... (vii)

Substituting the value of f in Eq. (vi),

$$\mu = \tan\theta - \frac{2}{3}\frac{g\sin\theta}{g\cos\theta} = \frac{1}{3}\tan\theta \quad \cdots \quad \cdots \quad \text{(viii)}$$

If μ gets a value less than this there will be partial slipping.

I. $\mu = \frac{1}{3}\tan 15 = \frac{1}{3} \times .2679 = .0893$

II. From Eq. (i),

$$f = (W\sin\theta - \mu W\cos\theta)\frac{g}{W}$$
$$= g(\sin\theta - \mu\cos\theta)$$
$$= 32.2\,(.2588 - .0893 \times .9659) = 5.56 \text{ ft. per sec. per sec.}$$
$$\alpha = 5.56 \div r = 5.56 \div 1 = 5.56 \text{ rad. per sec. per sec.}$$

The value of f can also be found out from the Eq. 178.

III. Linear velocity after rolling through 50 feet,

$v^2 = u^2 + 2fs = 2 \times 5.56 \times 50 = 556$

Therefore, $v = 23.6$ feet per sec.

and ω being equal to $\frac{v}{r} = 23.6$ rad. per sec.

Therefore, the kinetic energy,

$$K.E. = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$
$$= \frac{1}{2}\frac{8050}{32.2} \times 556 + \frac{1}{2}\frac{8050}{32.2} \times \frac{1}{2} \times 556$$
$$= \frac{1}{2} \times 250 \times 556\ (1 + \tfrac{1}{2})$$ — compare the form with that of the Eq. 176.

$$= 104200 \text{ ft. lbs.}$$

PRODUCT OF INERTIA

287. Product of Inertia. Second moment of the mass or the moment of inertia represents a property of the mass, but second moment of area is nothing but a quantity only, which is required in computations to solve some definite problems. Similarly, certain

expressions, such as, $\int dM\,xy$, $\int dA\,xy$, etc. which are nothing but quantities only just like the moment of inertia of area, appear in establishing the relation between the moments of inertia of masses and areas with reference to different axes.

These expressions may be said to be the second moments of masses and areas, as the case may be, with respect to two rectangular co-ordinate axes. They are better called *Products of Inertia* or *Moments of Deviation.*

Moments of inertia are always positive quantities being the products of masses or areas and the square of a linear distance, which may be positive or negative. The product of inertia, whereas, may be positive or negative depending on the signs of the two distances from the two rectangular axes respectively. It may be the case that the two distances may have the same sign, as well as, different signs with respect to the two axes from which the distances are measured.

288. Product of Inertia of an Area. The expression, $\int dA\,xy$, represents the product of inertia which is nothing but the limit of the sum of the products of the elementary areas into which the area may be conceived to be divided with the products of their distances from the two rectangular co-ordinate axes.

If the area be distributed in the four quadrants, then, the products of inertia of areas in the first and third quadrants will be positive and those in the second and fourth quadrants will be negative. The total product of inertia of the whole area will be the sum of these four. The product of inertia of an area is generally represented by the letter K. With reference to axes, K_{xy}, $K_{x1\,y1}$, etc., are used to denote products of inertia. The subscripts represent reference axes.

289. The Unit. The units of product of inertia are the same as those of the moments of inertia of areas—the symbols, $(\text{inches})^4$, $(\text{feet})^4$, etc. are used.

290. Product of Inertia with respect to parallel axes.

Let there be two pairs of rectangular co-ordinate axes X, Y and X_1, Y_1 in the plane of the area, parallel to each other and the last pair passing through the C. G. of the area (Fig. 272). Take an elementary area at distances of x_1 and y_1 and x_2 and y_2 from the X, Y

and X_1, Y_1 axes respectively. Let the distance between X and X_1 axes be y and that between Y and Y_1 be x. Then,

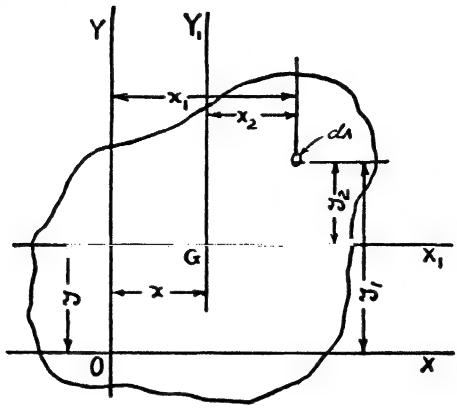


FIG. 272

$$K_{xy} \text{ of the area, } = \int dA\, x_1\, y_1$$

$$\text{and} \qquad K_{x1\,y1} = \int dA\, x_2\, y_2$$

The subscripts of K denote the pair of rectangular co-ordinate axes with respect to which the product of inertia is found out.

Now, $x_1 = x + x_2$

and $y_1 = y + y_2$

$$\begin{aligned} \therefore \quad K_{xy} &= \int dA\,(x + x_2)\,(y + y_2) \\ &= \int dA\, xy + \int dA\, xy_2 + \int dA\, yx_2 + \int dA\, x_2y_2 \\ &= \int dA\, xy + x \int dA\, y_2 + y \int dA\, x_2 + \int dA\, x_2y_2 \end{aligned}$$

Because the moment of area about an axis passing through C.G. is equal to zero, the second and the third terms of the above expression are equal to zero. Therefore,

$$K_{xy} = K_{x1\,y1} + A\,xy \qquad \cdots\cdots\cdots\cdots Eq.\ 181$$

That is, the product of inertia of an area with respect to a pair of rectangular co-ordinate axes is equal to the sum of the product of inertia with respect to another pair parallel to the previous one and passing through the C.G. of the area and the product of the area and the distances between the two pairs.

291. Deduction of Formulae for Product of Inertia of Areas with definite shapes.

Case 1. Rectangular area with respect to axes coinciding with two adjacent sides.

Take an elementary strip of width dx at a distance x from the Y-axis (Fig. 251). Let the length and breadth of the area be a and b respectively. Then the area of the elementary strip, $dA = b.dx$. Its moment about X-*axis* is $dA.y$, *i.e.*, $dA.\bar{y} = b.dx.\frac{b}{2} = \frac{b^2}{2}dx.$

Therefore, the product of inertia, *i.e.*, moment of this area about both the axes $= dA.xy = \frac{b^2}{2}\,x.dx.$

Hence, the product of inertia of the whole area,

$$K_{xy} = \int_0^a \frac{b^2}{2}\,x.dx = \frac{a^2 b^2}{4}$$

Case 2. Right angled triangle with respect to axes coinciding with the two sides—vertical and base. (Fig. 273)

Proceeding in the same way as in the previous problem, $dA = \frac{a(h-y)}{h}dy,$ where h is the altitude of the triangle. Therefore, the moment of dA about $OY = x.dA = \frac{a^2}{2h^2}(h-y)^2 dy,$ and the product of inertia with respect to OX and $OY = x.y.dA$

$$= \frac{a^2}{2h^2}(h-y)^2 y\,dy.$$

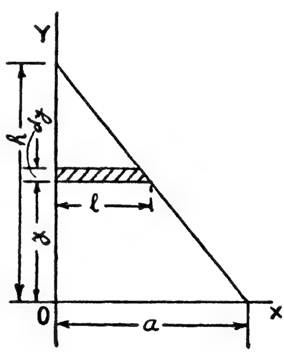


FIG. 273

Therefore, the product of inertia of the whole area,

$$K_{xy} = \frac{a^2}{2h^2}\int_0^h (h-y)^2.y.dy = \frac{a^2 h^2}{24}$$

The product of inertia of the area with respect to a pair of rectangular co-ordinate axes passing through the C.G. of the area and parallel to the previous pair, is found out from Eq. 181.

$$K_{xy} = K_{x1\,y1} + A\,xy$$

thus, $\dfrac{a^2 h^2}{24} = K_{x1\,y1} + \frac{1}{3}\,a.\ \frac{1}{3}\,b.\ \frac{1}{2}\,a.\,b$

$$K_{x1\,y1} = \frac{a^2 h^2}{24} - \frac{a^2 h^2}{18} = -\frac{a^2 h^2}{72}$$

Case 3. Quarter of a circular area with respect to two axes coinciding with the two radii bounding the area. (Fig. 274)

Method I. Cartesian co-ordinates.

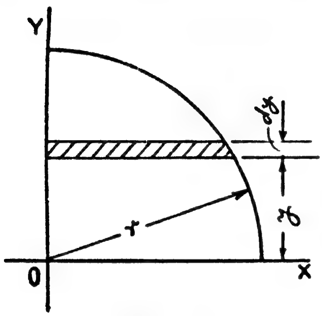


FIG. 274

Let r be the radius of the circle forming the area. Take an elementary strip of width dy at a distance y from the X-axis. Let its area be dA. Then, $dA = \sqrt{r^2 - y^2}.\,dy$. Its moment about $OY = x.\,dA = \frac{1}{2}\sqrt{r^2 - y^2}.\,dA$

$$= \tfrac{1}{2}\,(r^2 - y^2)\,dy.$$

Therefore, its product of inertia with respect to two axes,

$$= xy\,dA = \tfrac{1}{2}(r^2 - y^2)\,y\,ay$$

Hence, the product of inertia of the whole area,

$$K_{xy} = \tfrac{1}{2}\int_0^r (r^2 - y^2)\,y\,dy = \frac{r^4}{8}$$

Method II. Polar co-ordinates.

Let the elementary area be selected as shown (Fig. 275)

Then, $dA = r_1\,d\theta.\,dr_1$.

$\therefore \quad K_{xy} = \int xy\,dA$

$$= \int_0^r \int_0^{\frac{\pi}{2}} r_1\,cos\,\theta.\,r_1 \sin\theta.\,r_1\,d\theta.\,dr_1$$

$$= \int_0^r \int_0^{\frac{\pi}{2}} r_1^3\,dr_1.\,\tfrac{1}{2}\sin 2\,\theta.\,d\theta$$

$$= \tfrac{1}{2}\,.\,\frac{r^4}{4}\int^{\frac{\pi}{2}} \sin 2\,\theta.\,d\theta.$$

$$= \frac{r^4}{8}\left|-\frac{1}{2}\cos 2\theta\right|_{0}^{\frac{\pi}{2}} = \frac{r^4}{8}$$

(*b*) Product of inertia with respect to a pair of parallel axes passing through C.G. of the area.

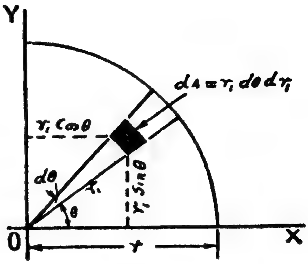


FIG. 275

The distance of the C.G. from the axes, $\bar{x} = \bar{y} = \frac{4r}{3\pi}$

(Art. 238—*Case V*)

From the relation, $K_{xy} = K_{x1\,y1} + A.\bar{x}\bar{y}$ (Art. 290)

$$\frac{r^4}{8} = K_{x1\,y1} + \frac{\pi r^2}{4}\left(\frac{4r}{3\pi}\right)^2$$

$$\therefore \quad K_{x1\,y1} = r^4\left(\frac{1}{8} - \frac{4}{9\pi}\right) = -.0165\ r^4.$$

292. Product of Inertia of an Area with respect to a pair of Axes, one of them being the Axis of Symmetry. (Fig. 276)

Let the area *ABC* be a portion of a circular area symmetrical about the *Y*-axis. Take any strip, as before, parallel to *X*-axis. If its area be dA, its moment about *Y*-axis, $dA.x = 0$, because the centre of gravity of the strip is on the *Y*-axis. Hence, its product of inertia $dA.xy$, will also be equal to zero.

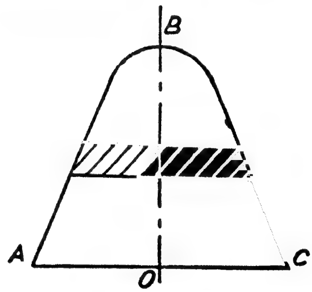


FIG. 276

Therefore, $K_{xy} = \int dA.xy = 0.$

If both the axes become the axes of symmetry both *x* and *y*, *i.e.*, $\bar{x}$ and $\bar{y}$ being equal to zero, $K_{xy} = 0$.

Thus, for each and every area symmetrically divided by at least one of the pair of axes the same argument holds good. *Therefore, the product of inertia of an area with respect to a pair of axes, at least one of them being the axis of symmetry, is equal to zero.*

293. Product of Inertia about different pairs of Rectangular Co-ordinate Axes having the same Origin. (Fig. 277)

Draw two pairs of rectangular co-ordinate axes, OX, OY and OX_1, OY_1 such that they have common origin O and the angle between the two pairs be α. Let I_x and I_y be the moments of inertia of the area about X and Y axes respectively and K_{xy} be its product of inertia

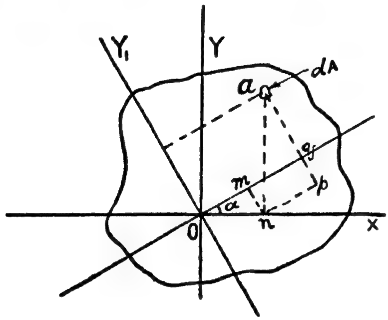


FIG. 277

about those axes. Also, let I_{x1} and I_{y1} be the moments of inertia of the area about X_1 and Y_1 axes respectively and $K_{x1\,y1}$ be the product of inertia about those axes. Take an elementary area dA at a, whose co-ordinates with respect to two pairs of axes are (x, y) and (x_1, y_1) respectively.

Then, $I_x = \int dA\, y^2$, $I_y = \int dA\, x^2$ and $K_{xy} = \int dA\, xy$

and, $I_{x1} = \int dA\, y_1^2$, $I_{y1} = \int dA\, x_1^2$ and $K_{x1\,y1} = \int dA\, x_1\, y_1$

But, $x_1 = Om + np = x \cos \alpha + y \sin \alpha$

and, $y_1 = ap - pq = y \cos \alpha - x \sin \alpha$

Substituting these values of x_1 and y_1 in I_{x1}, I_{y1} and $K_{x1\,y1}$

$$\begin{aligned} I_{x1} &= \int dA\, (y \cos \alpha - x \sin \alpha)^2 \\ &= \cos^2 \alpha \int dA\, y^2 + \sin^2 \alpha \int dA\, x^2 - 2 \cos \alpha \sin \alpha \int dA\, xy \\ &= I_x \cos^2 \alpha + I_y \sin^2 \alpha - 2\, K_{xy} \sin \alpha \cos \alpha \\ &= I_x \cos^2 \alpha + I_y \sin^2 \alpha - K_{xy} \sin 2\, \alpha \qquad \cdots\cdots\cdots Eq.\ 182 \end{aligned}$$

Similarly,

$$\begin{aligned} I_{y1} &= \int dA\, (x \cos \alpha + y \sin \alpha)^2 \\ &= I_x \sin^2 \alpha + I_y \cos^2 \alpha + K_{xy} \sin 2\, \alpha \qquad \cdots\cdots\cdots Eq.\ 183 \end{aligned}$$

And, $K_{x1\,y1} = \int dA\,(x \cos \alpha + y \sin \alpha)(y \cos \alpha - x \sin \alpha)$

$$= (I_x - I_y) \cos \alpha \sin \alpha + K_{xy} (\cos^2 \alpha - \sin^2 \alpha)$$

$$= \tfrac{1}{2} (I_x - I_y) \sin 2\alpha + K_{xy} \cos 2\alpha \qquad \cdots\cdots\cdots Eq.\ 184$$

The three relations established above are used for determining moments and products of inertia of areas with respect to different pairs of rectangular co-ordinate axes having the same origin.

294. Principal Axes and Principal Moments of Inertia. Through a point in an area any number of pairs of rectangular co-ordinate axes can be drawn. Of all these pairs there is a definite pair, about one of which the moment of inertia has the minimum value and about the other the maximum value. Such a pair of axes is called the *Principal Axes* with respect to the point through which they are drawn, and the moments of inertia about those two axes are called the *Principal Moments of Inertia.*

By differentiating the *Equation* 182 of the previous article,

$$\frac{d}{d\alpha} I_{x1} = 2 \cos \alpha \sin \alpha\ I_y - 2 \cos \alpha \sin \alpha\ I_x - 2 K_{xy} (\cos^2 \alpha - \sin^2 \alpha).$$

If the subscript of I represents the principal axis so that I_{x1} will be the least, the differentiation must be equal to zero, *i.e.*,

$$2 (I_x - I_y) \sin \alpha \cos \alpha + 2 K_{xy} (\cos^2 \alpha - \sin^2 \alpha) = 0 \qquad \cdots\cdots (i)$$

From which, $$\frac{2 K_{xy}}{I_y - I_x} = \frac{2 \sin \alpha \cos \alpha}{\cos^2 \alpha - \sin^2 \alpha} = \tan 2\alpha \qquad \cdots\cdots\cdots Eq.\ 185$$

Now, for the same value of $\tan 2\alpha$, there must be two values for 2α differing by 180°, *i.e.*, the values of α will differ by 90°. For the two values of α, I_{x1} will give two values—one of them will be maximum and the other minimum. The two values are for the two axes and, therefore, they are the values of I_{x1} and I_{y1} (the two axes differing by 90°).

295. Least Moment of Inertia and the Axis. From the theory of parallel axes it is evident that moment of inertia about any axis, $I = I_G +$ something, where I_G is the moment of inertia about an axis passing through C.G. and parallel to the axis referred to. Therefore, the axis about which the moment of inertia is the least must pass through the C.G. of an area. Product of inertia helps us to find out that definite axis and the moment of inertia about it, which is the

least. The purpose of determining the least moment of inertia of an area and the axis about which the moment of inertia is the least will be understood later on when higher technical mechanics on *Strength of Materials* will be studied.

As we are concerned with the least moment of inertia of an area and the corresponding axis, we shall, by the terms 'Principal Axes' and 'Principal Moments of Inertia' mean the definite pair of Principal axes that passes through the C.G. of the area and the moments of inertia about those two axes respectively.

If the value in equation (*i*) is put in the *Eq.* 184, then, $K_{x1\,y1} = 0$. That is, the product of inertia with respect to a pair of axes, at least one of which is a principal axis, is zero.

296. *An Axis of Symmetry of an Area is a Principal Axis.* Because the product of inertia with respect to a pair of axes, one of which is an axis of symmetry (which must pass through C.G. of the area) is zero (Art. 292), the axis of symmetry of an area is a principal axis. The other Principal axis must be in the plane of the area and at right angles to the axis of symmetry. It may pass through C.G. or may not, which depends on the position of the point of reference. However, we shall always take C.G. as the point of reference, the reason of which has been explained already.

297. *Sum of the moments of inertia about different pairs of rectangular co-ordinate axes through the same point is constant.*

By adding the *Eq.* 182 and *Eq.* 183 of Art. 293

$$I_{x1} + I_{y1} = I_x\,(\sin^2\alpha + \cos^2\alpha) + I_y\,(\sin^2\alpha + \cos^2\alpha)$$
$$= I_x + I_y$$

The subscripts, x_1 and y_1 represent any pair of axes. Therefore, for all pairs of axes it is true that the sum is equal to $I_x + I_y$, which is a constant quantity.

298. *By subtraction,*

$$I_{x1} - I_{y1} = (I_x - I_y)\cos 2\alpha - 2\,K_{xy}\sin 2\alpha \qquad \cdots\cdots\cdots Eq.\ 186$$

If I_x and I_y represent the moments of inertia about the principal axes, K_{xy} being equal to zero,

$$I_{x1} - I_{y1} = (I_x - I_y)\cos 2\alpha \qquad \cdots\cdots\cdots\cdots Eq.\ 187$$

299. Momental Ellipse or Ellipse of Inertia. It is a curve (ellipse) from which the moment of inertia of an area about an axis through a point in the plane can be easily calculated.

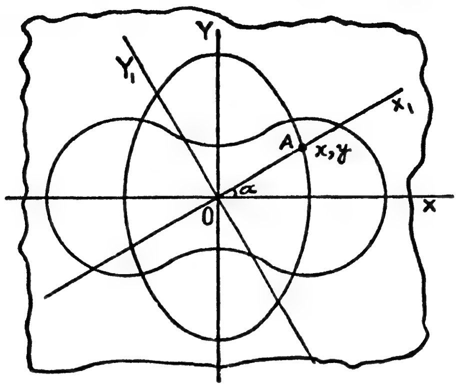


FIG. 278

Let XX and YY be the two principal axes of an area whose measure is a units (Fig. 278). Take a length $r_1 = \sqrt{\frac{a}{I_x}}$ along OX, and $r_2 = \sqrt{\frac{a}{I_y}}$ along OY. Draw an ellipse with r_1 and r_2 as semi-minor and semi-major axes respectively. Draw an axis $X_1 X_1$ through O, making an angle α with OX cutting the ellipse at A, whose co-ordinates are x and y. Let $OA = r$.

Then, taking $r = \sqrt{\frac{a}{I_{x1}}}$, $\frac{1}{r^2} = \frac{I_{x1}}{a} = \frac{I_x}{a}\cos^2 \alpha + \frac{I_y}{a}\sin^2 \alpha$

but, $\cos \alpha = \frac{x}{r}$ and $\sin \alpha = \frac{y}{r}$

$$\therefore \quad \frac{1}{r^2} = \frac{x^2}{r^2 . r_1^2} + \frac{y^2}{r^2 . r_2^2} \quad \text{or,} \quad \frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1, \quad \ldots\ldots\ldots(i)$$

which is an equation of an ellipse—A being any point on the curve and the value of OA, *i.e.*, r, the semi-diameter of the ellipse along $X_1 X_1$ being taken just in the same principle as was done in cases of r_1 and r_2—it is found that it satisfies the condition of an ellipse.

From this ellipse the moment of inertia can be calculated about any axis through O. The semi-diameters of the ellipse represent the

reciprocals of the radii of gyration of the area about the axes coinciding with the semi-diameters. I_{x1}, the moment of inertia about any axis $X_1 X_1$ through O, is equal to $\frac{a}{r^2}$. To actually draw the ellipse is possible only when the principal axes are known, *i.e.*, if the area is symmetrical about two rectangular co-ordinate axes, the principal axes being known, the ellipse can be easily drawn.

Cases of unsymmetrical areas will be treated in article 301.

300. Inertia Curve.

$$I_{x1} = I_x \cos^2 \alpha + I_y \sin^2 \alpha$$

If with different values of α the values of I_{x1} are found out or if with the help of the momental ellipse the values of I_{x1} about different axes $X_1 X_1$ are found out and the values are represented by the lengths along the axes measured from O and if a smooth curve is drawn through the tips of the lengths, the curve developed is called the *Inertia Curve.* The moment of inertia of the area about any axis can be readily read off from the length of the axis from O intercepted by the curve. (Fig. 278)

From the above discussion it is clear that the moments of inertia about two axes at equal angular distances from X or Y axis on either side of it are equal, because the semi-radii along those two axes are equal.

It is also clear that if $I_x = I_y$, *i.e.*, if the moments of inertia about two principal axes are equal, the ellipse will be a circle. In (*i*) Art. 299 if $r_1 = r_2$, the equation takes the form of, $x^2 + y^2 = r^2$, which is an equation of a circle. Hence, the moments of inertia about all the axes passing through the intersection of the principal axes are equal.

Again, it is evident that if the moments of inertia about more than two axes, not rectangular, are equal, the ellipse will be a circle and the moments of inertia about all the axes are equal.

In cases where the momental ellipse becomes a circle, the inertia curve will also be a circle. By choosing proper scales the same circle may represent momental ellipse as well as inertia curve.

301. Least Moment of Inertia in cases of Unsymmetrical Areas. The treatment of problems with symmetrical areas is simpler,

because the principal axes are known in those cases. But in cases where the areas are not symmetrical it is required to find out the principal axes with the help of which the momental ellipse and the inertia curve can be drawn. The necessity of determining the least moment of inertia will be clear when the question of designing the columns and struts arises. The method of finding out the principal axes and the least moment of inertia of such a section is given below : Take the section of an angle-iron. To find its principal axes through its C.G. and also to find the least moment of inertia of the area.

Procedure :—Find the C.G. of the area, O. Through O draw two rectangular co-ordinate axes, X_1 X_1 and Y_1 Y_1 parallel to the sides of the sectional area as shown in the diagram (Fig. 279). Solve out analytically the values of I_{x1}, I_{y1} and $K_{x1\,y1}$. Through the point O

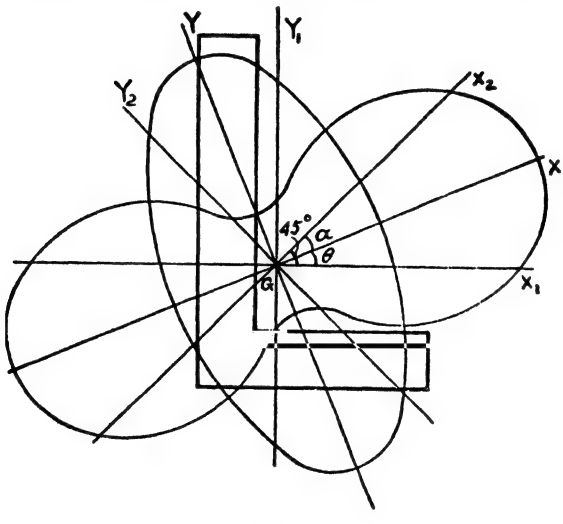


FIG. 279

draw another pair of rectangular axes $X_2\,X_2$ and $Y_2\,Y_2$, so that the axis $X_2\,X_2$ makes an angle, preferably 45°, with the axis $X_1\,X_1$. Next calculate out the values of I_{x2} with the help of the Eq. of Art. 293.

Then, $I_{y2} = I_{x1} + I_{y1} - I_{x2}$

Suppose XX and YY are the two principal axes through O, and suppose XX makes an angle θ with the axis X_1X_1 and δ with the axis X_2X_2.

$$\left.\begin{aligned} I_{x2} - I_{y2} &= (I_x - I_y)\cos 2\,\delta \\ \text{and } I_{x1} - I_{y1} &= (I_x - I_y)\cos 2\,\theta \end{aligned}\right\} \qquad (Eq.\ 187\ \text{Art. } 298)$$

$$\therefore \quad \frac{I_{x1} - I_{y1}}{I_{x2} - I_{y2}} = \frac{\cos 2\,\theta}{\cos 2\,\delta},$$

But, $\theta + \delta = 45°$

$$\therefore \quad \cos 2\,\delta = \cos 2\,(45 - \theta)$$
$$= \sin 2\,\theta$$

Hence,
$$\frac{I_{x1} - I_{y1}}{I_{x2} - I_{y2}} = \frac{\cos 2\,\theta}{\sin 2\,\theta}$$

$= \cot 2\,\theta$, from which θ can be determined.

Now, $I_x - I_y = \dfrac{I_{x1} - I_{y1}}{\cos 2\,\theta}$ $\quad$ (*Eq.* 187)

and $\quad I_x + I_y = I_{x1} + I_{y1}$

$$\therefore \quad I_x = \tfrac{1}{2}\left\{ I_{x1}\left(1 + \frac{1}{\cos 2\,\theta}\right) + I_{y1}\left(1 - \frac{1}{\cos 2\theta}\right)\right\}$$

$$\text{and} \quad I_y = \tfrac{1}{2}\left\{ I_{x1}\left(1 \quad \frac{1}{\cos 2\theta}\right) + I_{y1}\left(1 + \frac{1}{\cos 2\,\theta}\right)\right\}$$

Now, with reference to I_x and I_y, the moments of inertia about different axes are determined and with the values obtained the inertia curve can be drawn, which is as shown in Figure 279.

Alternative Method

With the help of the *Eq.* 185 the position of the principal axes can be determined.

The least moment of inertia,

$$\text{Either} = I_{x1}\cos^2\alpha + I_{y1}\sin^2\alpha - K_{x1\,y1}\sin 2\,\alpha$$
$$\text{or} = I_{y1}\cos^2\alpha + I_{x1}\sin^2\alpha + K_{x1\,y1}\sin 2\,\alpha$$

Note that the subscripts x_1 and y_1 represent here in this case the reference axes.

302. Mohr-Land Circle for determining the Moment of Inertia and Principal Axes.

First find the values of I_x, I_y and K_{xy} by the analytical method about two possible axes, X and Y preferably through C.G. as we always

want the least moment of inertia. Then, the values of I_{x1}, I_{y1} and $K_{x1\,y1}$ can be determined about any two axes, X_1 and Y_1, passing through the origin of the previous axes, graphically by the Mohr-Land Circle. Also the principal axes can be found out easily with the help of the circle which is explained below (Fig. 280).

I. Choose a scale for moment of inertia and product of inertia. Mark that both of them are in the same linear unit to the power four. Next, on the Y-axis of the rectangular co-ordinate axes X and Y measure OB and BE to represent I_x and I_y respectively in the scale chosen. Draw a circle on OE as diameter. Let C be its centre. From B draw a perpendicular BM to represent K_{xy}. If its value is positive, it is to be drawn to the right of the Y-axis, as shown. In case it is negative, it is to be drawn to the left.

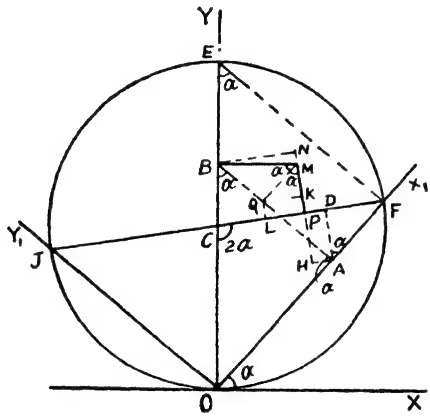


FIG. 280

From the origin, O, draw a straight line OX_1 cutting the circle at F and making an angle α with OX — α being the angle of obliquity of the axes about which the values are to be determined. Join FC and produce it to cut the circle at J. From M drop a perpendicular MP on FJ. Then, FP will represent I_{x1}, PJ will represent I_{y1} and MP will represent $K_{x1\,y1}$.

Proof. The dotted lines are the construction lines for the proof.

BA is parallel to EF ; MQ is perpendicular to BA ; AH, QK and BN are perpendiculars on Mp and MP produced ; AD and QL are perpendiculars on FJ. Equal angles are shown in the diagram (proofs are easy).

Now, $FP = LD + DF - LP$

But, $LD = AQ \sin(90 - \alpha) = AQ \cos\alpha$
$= (AB - BQ)\cos\alpha = (I_x \cos\alpha - K_{xy} \sin\alpha)\cos\alpha$

$DF = AF \sin\alpha = BE \sin^2\alpha = I_y \sin^2\alpha$

and, $LP = QK = QM \sin\alpha = BM \cos\alpha \sin\alpha$
$= K_{xy} \sin\alpha \cos\alpha$

(1) Hence, $FP = I_x \cos^2\alpha + I_y \sin^2\alpha - 2\,K_{xy} \sin\alpha \cos\alpha = I_{x1}$

(2) Again, $DJ = FJ - DF = OE - DF = I_x + I_y - I_{x1} = I_{y1}$

(3) $MP = HN - HP - MN = AB \cos(90 - \alpha)$
$- BE \sin\alpha \cos\alpha - BM \cos(180 - \alpha)$
$[HP = AD = AF \cos\alpha = BE \sin\alpha \cos\alpha]$

$\therefore\quad MP = I_x \sin\alpha \cos\alpha - I_y \sin\alpha \cos\alpha + K_{xy} \cos 2\,\alpha$
$= \frac{1}{2} I_x \sin 2\,\alpha - \frac{1}{2} I_y \sin 2\,\alpha + K_{xy} \cos 2\,\alpha$
$= \frac{1}{2}(I_x - I_y) \sin 2\,\alpha + K_{xy} \cos 2\,\alpha = K_{x1\,y1}$

II. From the Mohr-Land circle the principal axes and the principal moments of inertia can easily be determined as follows :

We know that the product of inertia about principal axes is zero. Therefore, the diagram should be drawn in such a way that MP in Fig. 280 vanishes.

Through M and C draw the diameter FJ, so that there is no existence of MP, *i.e.*, $K_{x1\,y1} = 0$. Join FO and JO. Now, FO and JO are the principal axes. MF and MJ represent I_{x1} and I_{y1} respectively. The angle between the principal axis OF, *i.e.*, OX_1 and the reference axis OX is α. (Fig. 281)

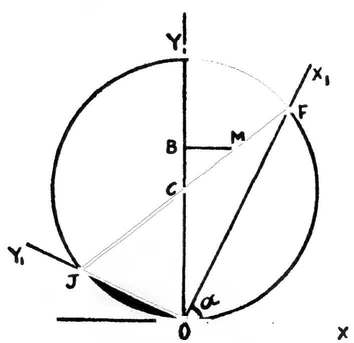


FIG. 281

Illus. Ex. 137. *Neglecting the filleting and rounding of edges of the rolled section (angle) as shown in Fig. 282-A, compute the moments of inertia and product of inertia with respect to the axes, X' and Y', and also with respect to two parallel axes passing through the C.G of the area. Determine the principal axes and the maximum and minimum moments of inertia.*

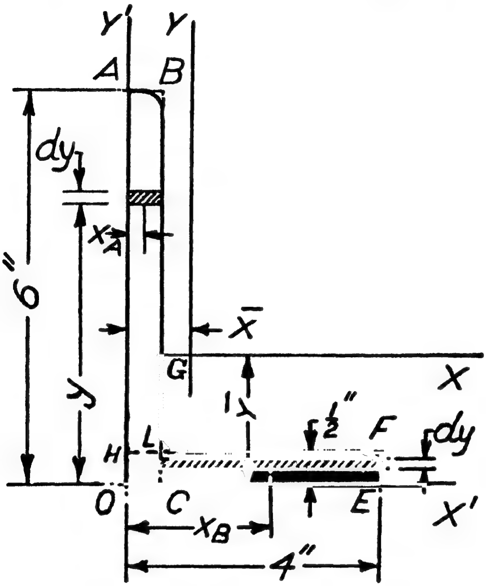


FIG. 282-A

I. C.G. of the area.

The area is divided into two rectangular areas *ABCO* and *CEFL*. Then, if a_1 and a_2 be the measures of the two areas,

$$x = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} = \frac{3 \times \frac{1}{4} + \frac{7}{4}\left(\frac{1}{2} + \frac{7}{4}\right)}{3 + \left(\frac{7}{2} \times \frac{1}{2}\right)}$$

$= .985$ inch.

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{3 \times 3 + \left(\frac{7}{4} \times \frac{1}{4}\right)}{3 + \left(\frac{7}{2} \times \frac{1}{2}\right)}$$

$= 1.99$ inches.

II. Moments of inertia and product of inertia with respect to X' and Y' axes.

Let I_{x1} and I_{x2} be the moments of inertia of the two areas with respect to X' axis.

Then, $I_{x1} = \frac{1}{3} . \frac{1}{2} . 6^3 = 36$ (in)^4

and, $I_{x2} = \frac{1}{3} . \frac{7}{2} . \frac{1}{8} = .145$ (in)^4

Therefore, $I_x' = 36 + .145 = 36.145$ (in)^4

Similarly, let I_{y1} and I_{y2} be the moments of inertia of the two areas (*ABLH* and *OEFH*) respectively with respect to the Y' axis.

$$\text{Then, } I_{y1} = \frac{5\frac{1}{2} \times (\frac{1}{2})^3}{3} = .23 \text{ (in)}^4$$

$$\text{and, } I_{y2} = \frac{\frac{1}{2} \times 4^3}{3} = 10.67 \text{ (in)}^4$$

Therefore, $I_y' = 10.67 + .23 = 10.9$ (in)^4

Transferring the axes through C.G.,

$$I_x = I_x' - A\,d^2 = 36.145 - \frac{19}{4}(1.99)^2 = 17.345 \text{ (in)}^4$$

$$I_y = I_y' - A\,d^2 = 10.9 - \frac{19}{4}(.985)^2 = 6.28 \text{ (in)}^4,$$

Where I_x and I_y are the moments of inertia of the area with respect to the two axes passing through the C.G. of the area and parallel to the given axes respectively.

Product of Inertia. Let $K_{x'y'}$ be the product of inertia of the area with respect to the axes X' and Y'. Dividing the area into rectangles *ABCO* and *CEFL*, the product of inertia,

$$K_{x'y'} = \int_0^6 \tfrac{1}{2}\,y \times \tfrac{1}{2}\,dy + \int_0^{.5} \tfrac{9}{4} \times y \times \tfrac{7}{2}\,dy$$

$$= \frac{36}{16} + \frac{63}{16} \times \frac{1}{4} = \frac{207}{64} = 3.23 \text{ (in)}^4$$

If, now, K_{xy} represents the product of inertia with respect to the axes through the centroid, X and Y, then,

$$K_{xy} = K_{x'y'} - A.\,\bar{x}.\,\bar{y} = 3.23 - (.985 \times 1.99 \times \tfrac{19}{4})$$
$$= -6.07 \text{ (in)}^4$$

III. Principal axes.

$$\tan 2\alpha = -\frac{2K_{xy}}{I_y - I_x} = -\frac{2 \times 6.07}{6.28 - 17.345} = 1.095$$

From which, $2\alpha = 47.5°$, and, $\alpha = 23.75°$

Hence, the principal axes are inclined at 23.75° and (90 + 23.75)° with the X-axis (anti-clockwise).

Maximum and Minimum moments of inertia.

$$\text{Maximum} = I_x \cos^2 \alpha + I_y \sin^2 \alpha - K_{xy} \sin 2\alpha$$
$$= 17.345 \times (.915)^2 + 6.28 \times (.402)^2 + 2 \times 6.07 \times .915 \times .402$$
$$= 20 \ (\text{in})^4$$

$$\text{Minimum} = I_x + I_y - \text{Maximum}$$
$$= 17.345 + 6.28 - 20 = 3.625 \ (\text{in})^4.$$

The principal axes and the moments of inertia can be solved graphically by the Mohr-Land circle as shown in Fig. 282-B which has been drawn in actual scale.

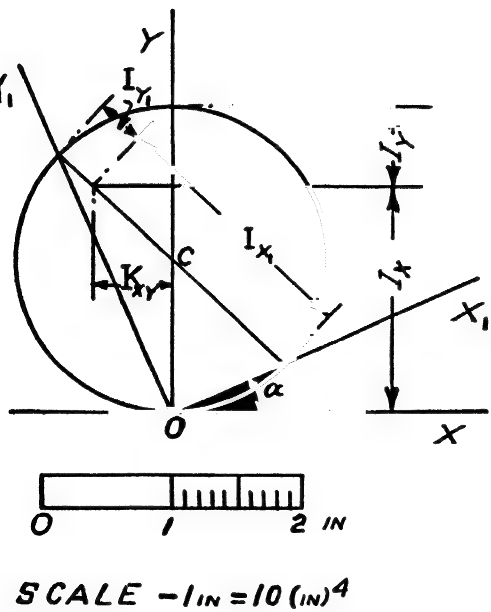


FIG. 282-B

According to the scale chosen — $\{ 1'' = 10 \ (\text{in})^4 \}$

$I_x = 17.34 \ (\text{in})^4$ *i.e.* $= 1.734''$

$I_y = 6.28 \ (\text{in})^4$ *i.e.* $= 0.628''$

$K_{xy} = -6.07 \ (\text{in})^4$ *i.e.* $= -.607''$

$I_{x1} = 2.009''$ *i.e.* $= 20.09 \ (\text{in})^4$

$I_{y1} = 0.360''$ *i.e.* $= 3.6 \ (\text{in})^4$

and $K_{x1y1} = 0$

303. Product of Inertia of Mass. The expressions $\int dM\, xy$, $\int dM.yz$, $\int dM.xz$ represent the product of inertia of a homogeneous mass with reference to different pairs of the three rectangular co-ordinate planes through a given point.

$$\text{Thus, } K_{xy} = \int dM._{xy}$$

$$K_{yz} = \int dM._{yz}$$

$$\text{and } K_{xz} = \int dM._{xz}$$

Treating in the same way as was done in cases of moments of inertia, the product of inertia of a mass may be determined from the algebraic sum of the products of inertia of the finite parts into which the mass may be conveniently divided.

304. The Unit. The unit in which the products of inertia of a mass is measured is the same with that of moments of inertia of a mass.

305. Plane of Symmetry. If one of the three rectangular co-ordinate planes of reference be a plane such that it divides the total volume into two symmetrical portions, then, the plane is called the plane of symmetry.

If one of the planes be a plane of symmetry, then, the product of inertia with respect to that plane and any one of the other two is equal to Zero, *i.e.*, $K_{xy} = 0$ and $K_{xz} = 0$, where X-plane is the plane of symmetry.

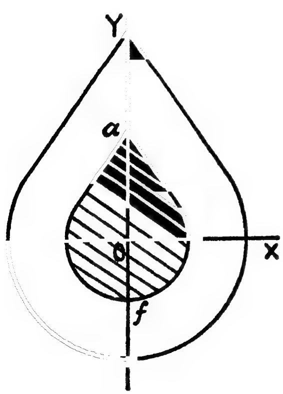


FIG. 283

Let a mass be symmetrical about the X-plane, Z-plane being represented by the plane of the paper (Fig. 283). If a section af be cut by a plane parallel to the Z-plane, then, the sectional area is symmetrical about the Y-axis and its product of inertia with respect to the X-axis and Y-axis is equal to $\int dA.xy$, and is equal to Zero.

If a slice with this area as surface and uniform thickness dz be considered and if the density of the mass be w, then, the product of inertia of

the slice with respect to X and Y planes,

$$= \int w.dz.dA.xy = w.dz \int dA.xy = 0$$

Now, if the whole body be divided into a large number of such slices, the product of inertia of each slice with respect to those two planes being zero, the product of inertia of the whole body with respect to those two planes is equal to zero, *i.e.*, $K_{xy} = 0$. Similarly, proceeding in the same way it can be shown that, $K_{xz} = 0$.

It is evident that if two of the three rectangular planes be the planes of symmetry, $K_{xy} = 0$, $K_{xz} = 0$, $K_{yz} = 0$.

306. Principal Axes, Principal Planes and Principal Moments of Inertia of a Mass. Moments of inertia about any axis through a given point can be determined following the methods adopted in cases of plane areas, in terms of moments and products of inertia of the mass with respect to three rectangular co-ordinate axes through a given point, when the angles between the given axis and the three rectangular axes are known. If I_x, I_y and I_z be the moments of inertia and I_{xy}, I_{xz} and I_{yz} be the products of inertia with respect to three rectangular axes and if α, β and λ be the angles between the given axis and X, Y and Z axes respectively, then, the moment of inertia about the given axis,

$$\begin{aligned} I = I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \lambda \\ - 2\, K_{xy} \cos \alpha \cos \beta - 2\, K_{xz} \cos \alpha \cos \lambda \\ - 2\, K_{yz} \cos \beta \cos \lambda \end{aligned}$$

Thus, determining the moments of inertia about different axes it is found that there is a definite group of three rectangular co-ordinate axes through a given point such that the moment of inertia about one of these three axes is maximum and about one it is minimum or at least about the remaining two the moments are equal but smaller than the former one. The three rectangular axes of this group are called the *Principal Axes* and the three planes at right angles to these axes respectively are called the *Principal Planes* and the moments of inertia about these principal axes or planes are called the *Principal Moments of Inertia.*

Following the same arguments as were advanced in the case of an area it is evident that planes of symmetry are the principal planes and hence, if the mass be such that two of its planes are the planes of symmetry, then, the products of inertia with respect to three rectangular

planes, taking two at a time, are equal to zero. Therefore, $I = I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \lambda$.

The geometric representation of the moments of inertia about different axes through a point becomes an ellipsoid, which is known as *Momental Ellipsoid or Ellipsoid of Inertia.*

307. *There is an analogy between the linear and angular motions, which is shown in the following table* :

LINEAR MOTION	ANGULAR MOTION
1 s = linear displacement	θ = angular displacement
2. M = mass, $\frac{W}{g}$	I = moment of Inertia, $\int \frac{W}{g} x^2 = \frac{W}{g} k^2$
3 v = linear velocity, in case of const. velocity $= \frac{s}{t}$, in case of variable velocity $= \frac{ds}{dt}$	ω = angular velocity, in case of const. velocity $= \frac{\theta}{t}$, and in case of variable velocity $= \frac{d\theta}{dt}$
4. f = linear acceleration, in case of uniform acc. $= \frac{v}{t}$, and in case of variable acc. $= \frac{dv}{dt} = \frac{d^2s}{dt^2}$	α = angular acceleration, in case of uniform acc. $= \frac{\omega}{t}$, and in case of variable acceleration $= \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
5. P = effective force $= M.f$	T = effective torque $= I.\alpha$
6. Linear Momentum $= M.v$	Angular Momentum $= I.\omega$
7. Impulse $= P.t$	Impulse of a torque $= T.t$
8 Kinetic Energy, $K.E. = \frac{1}{2} M v^2$	$K.E. = \frac{1}{2} I \omega^2$
9. Work $= P.s$	Work $= T.\theta$
If v be the final velocity, u be the initial velocity and f be the uniform acceleration, $v = u \pm f t$ $s = u t \pm \frac{1}{2} f t^2$ $v^2 = u^2 \pm 2 f s$	If ω be the final angular velocity, ω_i be the initial velocity and α be the uniform angular acceleration, $\omega = \omega_i \pm \alpha t$ $\theta = \omega_i t \pm \frac{1}{2} \alpha t^2$ $\omega^2 = \omega_i^2 \pm 2 \alpha \theta$

PROBLEMS

327. Find the moment of inertia of a rectangular area, $8'' \times 6''$, about an axis, (1) coinciding with the smaller side, (2) coinciding with the bigger side. Also, determine the moment of inertia of the area about an axis passing through the C.G. of the area and parallel to, firstly, to the smaller side, and secondly, to the bigger side. Compare the results and mention the axis about which the moment of inertia is the least.

Ans. 1024 $(\text{in})^4$, 576 $(\text{in})^4$, 256 $(\text{in})^4$, 144 $(\text{in})^4$.

328. If from the above area, $6'' \times 4\frac{1}{2}''$ be removed such that the two figures have common diagonals, find the moment of inertia of the portion left about an axis coinciding with the smaller side of the bigger area. Find also the moment of inertia about an axis passing through the C.G. of the area and parallel to the above axis, and also about an axis coinciding with the shorter side of the area removed. What is the radius of gyration in each case?

Ans. 511 $(\text{in})^4$, 175 $(\text{in})^4$, 364 $(\text{in})^4$
4.95 ins., 2.9 ins., 4.16 ins.

329. Find the moment of inertia of an equilateral triangular area, ABC, whose sides are 4 inches long each and from which a similar area, abc, of 3 inches side is removed, about the base AB of the bigger triangle. The two areas have common medians. Find also the radius of gyration with respect to AB.

Ans. 7.85 $(\text{in})^4$, 1.496 in.

330. Compute the moments of inertia of an angle-iron section like that in Fig. 199, where the sides are 5 and 3 inches respectively, and the width throughout is .5 inch, about the X and Y axes. Also, determine the moments of inertia about parallel axes passing through the centroid of the area.

Ans. $I_x = 21.458\ (\text{in})^4$, $I_y = 8.037\ (\text{in})^4$
$I_{Gx} = 10.458\ (\text{in})^4$, $I_{Gy} = 4\ 897\ (\text{in})^4$
$\bar{x} = .855$ in., $\bar{y} = 1.656$ ins.

331. Find the moment of inertia of the area of the angle-section shown in Fig. 199 about X and Y axes.

Ans. 54[illegible] $(\text{in})^4$, 7.49 $(\text{in})^4$

332. Determine the moment of inertia of a section of a circular shaft 5" in diameter, about a diameter of the section. If the shaft would have been a hollow one, the inner diameter being 3 inches, what would be the moment of inertia about the same axis? Find the radii of gyration in both the cases.

Ans. 29.97 $(\text{in})^4$, 26.7 $(\text{in})^4$
1.25 ins., 1.45 ins.

333. What is the moment of inertia of a T-section of which the head is 4 inches long and the tail 5 inches, the widths of both the portions are the same and equal to $\frac{1}{2}$ inch, about the horizontal axis passing through the centroid of the area?

Ans. $\bar{x} = 0$, $\bar{y} = 1.583$ in., $I_{Gx} = 4.3\ (\text{in})^4$

334. Compute the moments of inertia of a channel section ($\sqcup$), about the horizontal and vertical axes passing through the C.G. of the area. The length of the flanges is 9 inches each, the length of the rib is 9 inches and the average thickness of the flanges, taken as uniform, is .5 inch and that of the rib is .4 inch. *Ans.* $\bar{x} = 0$, $y = .6$ in.
$I_{Gx} = 10.68$ (in)^4, $I_{Gy} = 103.35$ (in)^4

335. Determine the moments of inertia of the area given in Fig. 243 about two axes passing through the C.G. of the area and parallel to the common diameter and at right angles to it.
Ans. $x = 0$, $y = 2.3$ ins
$I_{Gx} = 68.7$ (in)^4, $I_{Gy} = 10.6$ (in)^4

336. Steel weighs 490 lbs. per cubic foot. A cone made of steel is 16 feet in height, from the apex end of which a portion, 8 inches in altitude, is cut off. Compute the moment of inertia of the frustum about its geometric axis.
Ans. .3856 unit or slug (ft)^2.

337. If in the frustum of the previous problem a concentric hole, 4 inches in diameter, is cut throughout along the axis, find the moment of inertia about the same axis. *Ans.* .3734 unit or slug (ft)^2.

338. Find the moment of inertia of the body shown in full section in Fig. 235—from the frustum of a cone a co-axial cylindrical portion has been removed—about the axis of the body (Y-axis).

339. Determine the moment of inertia of a cast iron cylinder weighing 268 $\frac{1}{3}$ lbs. and having a diameter of 4 feet, with respect to an element of its curved surface. *Ans.* 50 slug (ft)^2.

340. Determine the moment of inertia of a body consisting of a cylindrical portion and a conical one, shown in full section in Fig. 237, made of cast iron about the axis of the body (X-axis) and about an axis coinciding with a diameter of the plane surface at the end of the cylindrical portion (*i.e.*, about Y-axis).

341. Determine the radius of gyration of the fly-wheel represented by a half sectional view in Fig. 244, with respect to the axis of the wheel (X-axis). The moment of inertia is 32.2 slug (ft)^2. *Ans.* 14.91 inches.

342. Compute the moment of inertia of a straight uniform metallic rod weighing 100 lbs. about an axis perpendicular to the centre line of the rod and at $\frac{1}{8}$ th. of its length from one end. The length of the rod is 10 feet.
Ans. 44.5 slug (ft)^2.

343. If the fly-wheel in problem 313 has a speed of 300 r.p.m. and a radius of gyration equal to 14.9 inches with respect to the axis of rotation,

determine the moment of inertia of the wheel, about that axis. If a brake is applied at the rim surface and the wheel comes to rest after 100 complete turns, find the brake force tangential to the path of rotation. Neglect the bearing friction. *Ans.* 33.2 slug $(ft)^2$, 17.4 lbs.

344. In a laboratory experiment for determining the moment of inertia and radius of gyration of a fly-wheel, the weight of the rotating mass is given as 322 lbs. and it is found that the wheel is fixed on a 3-in. shaft.

A cord is wrapped round the shaft—one end is loosely fitted with it and from the other end a weight of 50 lbs. is suspended. The suspending weight starting from rest can fall vertically creating rotational motion in the wheel.

The average reading of 10 such operations gives that to fall 4 feet the weight requires 8 seconds. If the bearing resistance is taken as constant and equal to 6 lbs., find the moment of inertia of the rotating body and its radius of gyration with respect to the axis of rotation.

Ans. 4.225 slug $(ft)^2$ 7.8 inches.

345. A 5-ft. fly-wheel, weighing 3220 lbs., has a radius of gyration of 20 inches about its axis of rotation. It is subjected to a torque of 2400 in. lbs. Determine the tangential linear acceleration of a point on the surface of the rim.

346. A torque of 2500 lb. ft. is applied on a fly-wheel at rest, weighing 8050 lbs. In what time will the fly-wheel reach a speed of 120 r.p.m. if the radius of gyration be 2.5 feet? What is the work done by this time?

Ans. 7.76 secs., 123300 ft. lbs.

347. If a fly-wheel weighing 1610 lbs. rotates at a speed of 600 r.p.m. about an axis which is .04 inch off-centre, compute the effective radial force. What is the tangential effective force in this case? *Ans.* 658 lbs., nil.

348. The speed of a fly-wheel rotating at 98 r.p.m. is accelerated to a speed of 102 r.p.m. Find the moment of inertia of the fly-wheel, if the work done to change the speed is 100000 ft. lbs. At what speed of the wheel is the kinetic energy equal to the energy required to change the speed?

Ans. 22800 slug $(ft)^2$, 28.28 r.p.m.

349. A thin uniform straight rod, 327 c.m. in length, is suspended from a horizontal axis at one of its end, about which it can rotate freely. What initial linear velocity imparted to the free end of the rod enables the rod to reach the horizontal position? *Ans.* 981 c.m. per second.

350. If the rod of the previous problem is cut to a measure of 5 feet and held in a vertical position over the axis and the motion of the rod is set by a little disturbance, what will be the linear velocity of the free end when it comes vertically down the axis? *Ans.* 31.08 feet per second.

351. A pulley of 4 feet diameter weighs 644 lbs. It is mounted on a shaft about which it can rotate most freely. A cord is wound around it. If

a constant pull of 10 lbs. is maintained in the cord, find the angular acceleration of the pulley and the length of the cord unwound in 8 seconds starting from rest. *Ans.* .5 rad. per sec. per sec.
16 feet.

352. If instead of the constant pull a weight of 10 lbs. is suspended from the end of the cord in the previous problem, what will be the angular acceleration of the pulley motion and what is the linear acceleration of the weight suspended with which it is falling down? What is the tension in the cord?
Ans. .484 rad. per sec. per sec.
.968 ft. per sec. per sec.
9.68 lbs.

353. If a thin straight rod rotates with an accelerated motion about an axis at one end at right angles to the axis of the rod, prove that the effective force acts at a distance of two-thirds of its length from the axis.

354. A steel disc of 4 feet diameter and 3 inches thickness is constrained to rotate about a horizontal axis at right angles to the plane surface of the disc and coinciding with an element of the curved surface. It is held at rest in a position so that a diameter of the plane surface makes an angle of 30° with the horizontal direction upwards. It is then released so that it can rotate due to gravity alone. Find the magnitudes of the radial and tangential components of the reaction at the support at the time of release. Also determine the reactions when the disc has moved through 60° only. Steel weighs 490 lbs. per cu. ft. *Ans.* 770 lbs., 444.5 lbs., 2833.2 lbs. the same.

355. If the axis of rotation of the disc in the previous problem be tangential to the circumferential surface of the disc and passes through the middle point of an element of the curved surface and also be at right angles to the geometric axis of the disc, determine the radial and tangential effective forces when it has turned through 60° only from the position of rest as explained in the previous problem. Locate the point of application of the forces.
Ans. 2464 lbs., 1067 lbs., 2.5 ft. from the axis.

356. If a shaft of 6 inches diameter coasts down from 120 r.p.m. determine the number of revolutions it makes before it comes to rest. Coefficient of friction in the bearing is .02. *Ans.* 2.43 revolutions.

357. A straight thin rod 6 feet long is suspended from a horizontal axis through one end. Determine the number of oscillations it makes per minute when it is made to do so. Can you locate any other point in the rod through which the axis may be made to pass without changing the periodic time of the oscillation? *Ans.* 27 1, 2 ft. from the end.

358. If a thin straight rod has a periodic time of 1 second when oscillating about a horizontal axis through one end, determine the length of the rod.
Ans. 1.224 ft.

• 359. A thin straight rod oscillates 30 times per minute when suspended from a horizontal axis and made to oscillate about it. Determine the length of the rod if the axis passes through a point 3 inches from the end of the rod.
Ans. 14.2 feet.

• 360. Locate the centre of percussion of a plank of wood, 3 feet long, 2 inches thick and of uniform width. The plank is constrained to rotate about an axis on a plane of symmetry parallel to the wider face of the plank and passing through and coinciding with one thinner and smaller end of the plank. *Ans.* 2.006 ft.

• 361. In an experiment to find out the moment of inertia of a fly-wheel about its central axis, the weight of the wheel is 1610 lbs. It is placed hanging on a horizontal knife-edge at the inner rim surface which is at a distance of 3 feet from the axis. If the condition of equilibrium of the wheel is disturbed and if the wheel is found to oscillate 30 times per minute, determine the moment of inertia of the wheel about an axis passing through the knife-edge and parallel to the central axis and also about the central axis. What are the radii of gyration in the two cases respectively?

• 362. A compound pendulum is made up of a $\frac{1}{2}$ inch thick circular disc of steel of 6 inches diameter and a slender straight rod 3 feet in length, weighing 2 lbs. The pendulum oscillates about a horizontal axis at right angles to the plane surface of the disc passing through a point at a distance of 3 inches from one end of the rod. The other end of the rod reaches the centre of the disc surface. Find the moment of inertia and the radius of gyration of the pendulum with respect to the axis of rotation. What is the periodic time of the pendulum? Take the weight of the steel as 489 lbs. per cu. ft. *Ans.* 1.087 slug $(\text{ft})^2$, 2.415 ft., 2.475 secs.

• 363. A mass is suspended by means of a wire so that the end attached to the mass is fixed to the C.G. of the mass. A torque of 1 lb. in. twists the wire through 10°. Compute the moment of inertia of the mass neglecting the consideration of the mass of the wire. The suspended mass oscillates 120 times per minute due to the effect of the torque.
Ans. .006 slug $(\text{ft})^2$

364. A shaft 4 inches in diameter is placed horizontally and fixed at one end. Near the other end there is a ball bearing which acts as a support to check bending. At the free end a fly-wheel weighing 3220 lbs. is fixed and a torque of 500 lb. ft. is applied to twist the shaft through 1 degree and the shaft is let free. If the weight of the shaft is 250 lbs. and the radius of gyration of the fly-wheel is 3.5 feet, determine the period of vibration of the fly-wheel.

• 365. A rolling steel ball weighing 32.2 lbs. gets a rigid slope of 15° while it is moving with a constant speed of 2 feet per second. After rolling down

the slope for 50 ft. it strikes a spring resistance directly. If the characteristic of the spring is such that a load of 100 lbs. compresses it by 1 inch, and if the compression is proportional to the load applied, determine the displacement of the resistance before the ball reaches zero velocity. Steel weighs 490 lbs. per cu. ft. *Ans.* 3.737 inches.

366. A motor truck has four wheels, the combined weight of which is 10% of the total weight of the truck. If the total *K.E.* of the truck at a definite instant while running is $\frac{1}{10}$ time greater than the energy due to the motion of translation only and if the extreme diameter of the wheels is 3 feet, compute the radius of gyration of the wheels with respect to the axis of rotation. *Ans.* 12 inches.

367. A solid roller 4 feet in diameter weighs 3220 lbs. Its pulling arrangement measures 5 feet in length from the axis of roller. The hand grip of the pullers is at a height of 5 feet above the ground. If the roller starting from rest gets a speed of 4 feet per second after moving for 8 feet, determine the pulling force neglecting the bearing resistance.

Ans. 187.5 lbs.

368. Two wheels of radius $R = 1$ ft. are rigidly connected at the two ends of a drum of radius $r = .5$ ft., so that they have common axis of rotation. If the whole system which weighs 161 lbs. be drawn on a horizontal plane with a force $P = 20$ lbs. applied at the end of a cord wrapped round the surface of the drum, find the linear velocity of the axis of the body when it rolls through a distance of 20 feet. Assume that there is no slip and the pulling force is never off-centre and the radius of gyration square is .5 (in)^2 (Fig. 284).

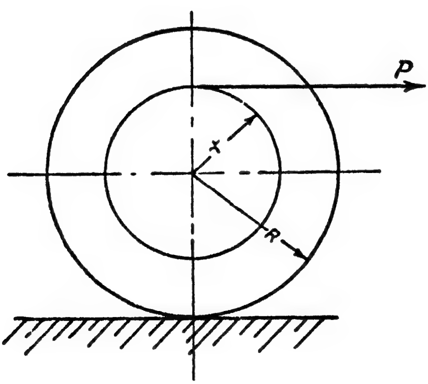


FIG. 284

Ans. 12.65 ft. per second.

369. If in the previous problem force be applied by suspending a mass of 20 lbs. at the free end of the cord which passes over a frictionless pulley so that the cord remains horizontal, find the velocity.

Ans. 11.62 ft. per second.

370. If in the problem 368 the wrapping of the cord be reversed and the pulling cord be tangential to the drum surface at the lowest point, determine the velocity of the central axis. Is there any difference in this case regarding the direction of rolling? *Ans.* 7.3 ft. per sec. No, the direction is clockwise.

• 371. A drum for a lifting machine weighs 200 lbs. and has a diameter of 4 feet. If a weight of 50 lbs. is suspended from a cord wrapped round the drum and the system is let free from rest, compute the velocity of the descending weight after 2 secs. and the tension in the cord. Neglect the bearing resistances. Solve with the help of D'Alembert's principle.

Ans. 28.6 ft. per sec., 35.8 lbs.

372. Discuss about the linear and angular speeds of spherical bodies of different diameters when they starting from rest at the top of an inclined plane roll down the plane without slipping.

Ans. $v =$ the same, ω varies with r.

• 373. If a homogeneous spherical body of diameter d or radius r rolls down an inclined plane of 30° with the horizontal without any slip starting from rest, find the linear velocity of the body when it has come down for 40 feet along the inclination. What is the kinetic energy of the body at that instant if $r = 1$ foot and the weight of the body is 161 lbs?

Ans. 30.33 ft./sec., 3221 ft. lbs.

374. If the spherical body of the previous problem be a hollow one and the ratio between the internal and the external diameters be 1 : 2, determine the linear velocity of the body under the same conditions.

Ans. 29.88 ft./sec.

• 375. A homogeneous spherical body whose diameter is 2 feet rolls down an inclined plane of 30° slope to reach a horizontal plane at the foot along the shortest course which is 100 feet long. The coefficient of friction between the body and the plane is .5. The inclined plane is greased for a depth of 40 feet measured from the bottom and the coefficient of friction is reduced thereby by 90%. The body rolls down the upper portion without slipping and rolls down and slips along the remaining portion. Determine the linear velocity of the centroid of the body when it reaches the horizontal plane. *Ans.* 50.56 ft./sec.

• 376. If the body of the previous problem weighs 322 lbs., what is the kinetic energy of the body when it reaches the horizontal plane?

Ans. 15585 ft. lbs.

377. Determine the moments of inertia of the Z-section (Fig. 285) about OX_1, OY_1 and about the two axes X and Y parallel to these two axes respectively as shown. Determine the least moment of inertia of the section and the axis about which the moment of inertia is the least, both analytically as well as graphically.

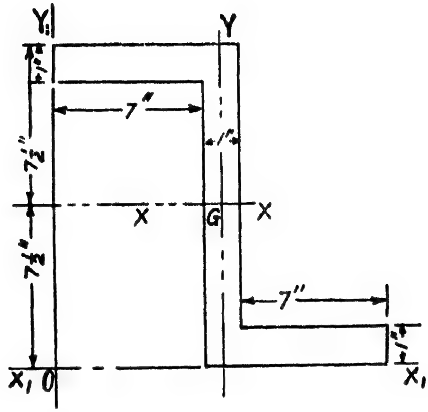


FIG. 285

Ans. 3602.33 (in)^4, 1913.5 (in)^4,
971.3 (in)^4, 282.5 (in)^4,
28 (in)^4, 27.5° with GX axis.

378. Determine the maximum and the minimum moment of inertia of the T-section in prob. 333. Construct the inertia curve and momental ellipse for the section.

CHAPTER XI

GRAPHICAL STATICS

308. In this chapter only graphical methods will be adopted in solving different kinds of problems of Statics. There are cases where graphical methods are more advantageous and simpler than analytical methods in solving problems. A few cases, of course, of concurrent forces only have already been treated graphically in Chapter VII. But elaborate discussion in the subject is required to tackle with different types of problems. A new system of notations for the forces is used and, therefore, a proper discussion of the system will be made before we proceed with the subject.

309. Bow's Notation. The notations adopted uptil now to represent forces are single letters, such as, *P*, *Q*, *R*, etc., or double letters, such as, *AB*, *BC*, *CD*, etc., remaining at the two terminii of the vectors. But, in this method, the letters, *A*, *B*, *C*, *D*, etc., are used to denote different spaces separated by the lines of action of the forces, and not the forces. Here also two letters, *AB*, *BC*, *CD*, etc., are used to denote forces, but their positions are not at the two ends of the vectors, they remain on the two sides of the vectors. For example, a force *AB* (Fig. 286) denotes the force, the line of action of which demarcates the two spaces *A* and *B*. This kind of notation of a force is named *Bow's Notation.* Generally the capital letters are used for these notations. It is to be noted that small letters have been used and will be always used in the vector diagrams.

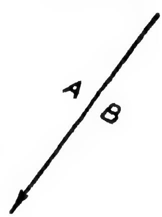


FIG. 286

310. Space Diagram and Force or Vector Diagram. In Fig. 287-I, a balanced system of five concurrent forces, *AB*, *BC*, *CD*, *DE* and *EA* (*i.e.*, *P*, *Q*, *R*, *S* and *T*) are shown by drawing their lines of action in proper positions in space. This diagram is called the *Space Diagram.* The diagram representing the geometric addition (Fig. 287-II) is called the *Vector* or *Force Diagram.*

311. Funicular or Link Polygon. It is clear that in cases of concurrent forces, when they do not form a balanced system, the

vector diagram is quite sufficient to obtain the resultant or the equilibrant in magnitude and in proper position in space, because we know that the lines of action of the forces will meet at a common point. But, in cases of non-concurrent forces the position in space of the

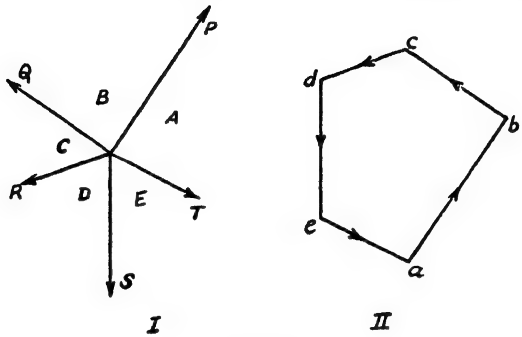


FIG. 287

resultant or the equilibrant cannot be determined from the vector diagram only, though the magnitude and direction are obtained from it. The reason is that we do not get any point through which its line of action can pass. We can, of course, by the principle of moments find the proper position in space of the force analytically. The graphical method of determining the proper position of the resultant or the equilibrant in space is as follows :

Let four co-planer non-concurrent forces, *AB*, *BC*, *CD* and *DE* act on a rigid body, which are drawn in their proper position in space (Fig. 288). Draw the vector diagram *abcde*. Then, *ea* is the equilibrant and *ae* is the resultant. Though the direction and the magnitude

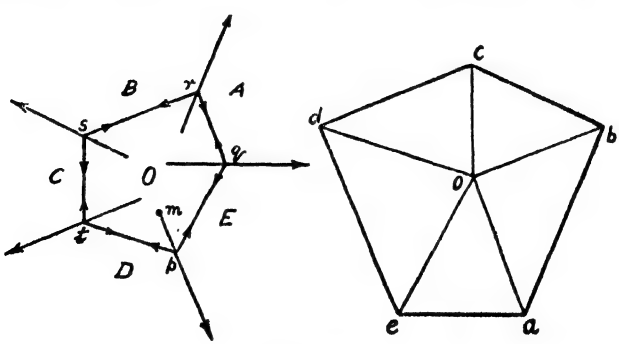


FIG. 288

are known from the diagram, yet the position in space is not known. Now, the method of determining the position in space is to take a point o in or outside the force diagram. Then, join oa, ob, oc, od and oe. Take a point r on the line of action of the force AB. From r draw a straight line rs parallel to ob to cut the line of action of the force BC at s. Mark that the line rs, linking AB with BC, is parallel to the line that joins o with the intersection of the vectors ab and bc. Thus, draw ~~rt~~ st, tp parallel respectively to oc and od. Next, from p draw pq parallel to oe and from r, rq parallel to oa cutting each other at q. Then, q is a point on the line of action of the equilibrant or the resultant. Draw $EA \parallel ea$ and equal to it through q.

Proof. The closed polygon, $rstpq$, thus obtained having their vertices on the lines of action of the forces is called a *Funicular Polygon* or *Link Polygon*. If a point O be taken within the funicular polygon, the sides of the polygon may be named OA, OB, OC, OD and OE respectively. Mark that $OA \parallel oa$, $OB \parallel ob$, $OC \parallel oc$ and so on. The polygon is thus named because these lines, OA, OB, etc., form a linkage amongst the forces. If instead of acting on a rigid body the forces act at the joints of a rigid structure, say $rstpq$, made up of five members, OA, OB, OC, etc., the effect will remain the same. Under the condition of equilibrium the force at each joint must be balanced by the forces acting along the two members forming the joint. From the directions of these two forces (as if they are pulling the joint) it is clear that the members are in tension. However, it is to be marked here that at each joint there are three forces in equilibrium. From the directions of the forces acting through the members it is evident that each member possesses two arrow-heads in opposite directions, aiming at each other, at the two ends indicating the directions of forces acting at the joints through the member (Fig. 288). After completing the funicular polygon from the force diagram it is found that there are two forces acting through the members OA and OE in the directions away from the point of intersection of the two members, which must be balanced by a third force to maintain the equilibrium of the joint q. The equilibrant of the four forces being ea, the force ea will be acting through the point of intersection of pq and rq.

The funicular polygon drawn is said to be the funicular polygon of the five forces, AB, BC, CD, DE and EA with respect to the point o in the force diagram. Thus, the funicular polygon for a system of

forces in equilibrium may be innumerable as the position of o may be innumerable. The point o is called the *Pole of the Funicular Polygon.*

312. Condition of equilibrium in graphical methods. In case of concurrent forces the condition is that either the vector diagram will be a closed figure, or, the sum of the horizontal and vertical components will be individually zero, or, the sum of the moments of the forces about any point in the plane of the lines of action of the forces must be zero. It is to be marked, which has already been said, that it is quite sufficient to prove that the vector diagram is a closed figure —no further proof is required. But in case of non-concurrent forces the conditions are that either of the first two conditions must be satisfied and the third condition also must be satisfied.

In graphical treatment with the non-concurrent forces it is found that two diagrams are required to be drawn to determine the equilibrant or the resultant fully—one, the force polygon, and, the other, the funicular polygon. Both of them are closed figures. The conditions, therefore, according to the graphical treatment are—(1) force or vector diagram is a closed figure, and, (2) funicular polygon is a closed figure. Because, though the force diagram of a system of forces may be a closed figure the funicular polygon may not be closed as is shown from the following example. In Fig. 289 if the forces

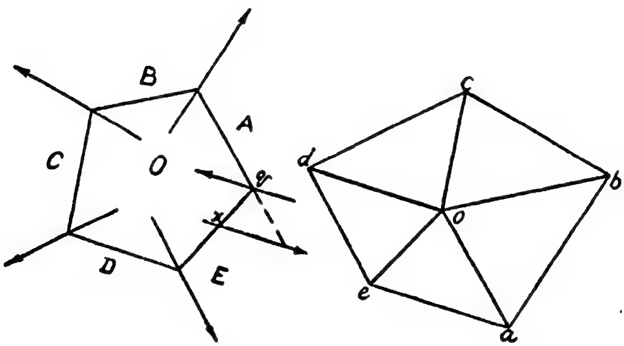


FIG. 289

AB, *BC*, *CD*, *DE* and *EA* be placed in space in such a way that the force *EA* instead of acting through q, acts through x, then, the force diagram will be a closed figure no doubt, but it is evident from the diagram that the funicular polygon is not closed and hence, the system cannot be in equilibrium, because the effect of the four forces *AB*,

BC, *CD* and *DE* is to create a resultant *ae* at *q*. Thus, if *EA* acts at *x* and a force equal to *ae* acts at *q*, then, the two equal and opposite forces will form a couple and the condition of equilibrium of the system will be disturbed. Therefore, to maintain equilibrium the force *EA* must act through *q* and the funicular polygon of the forces must be a closed figure.

313. Parallel forces. In case of parallel forces it is evident that in drawing the force diagram all the forces represented by vectors lie in a straight line. If the last vector ends at the starting point the figure is taken as a closed one and the forces are said to be in equilibrium. In case of parallel forces acting on a rigid body the treatment is just similar to the general non-concurrent forces.

Let *AB*, *BC*, *CD* and *DE* be four parallel forces acting on a rigid body. To find the equilibrant of the system. Draw the vector diagram *abcde* (Fig. 290). Then, *ea* is the equilibrant representing the magnitude and direction. Now, to find out its position in space,

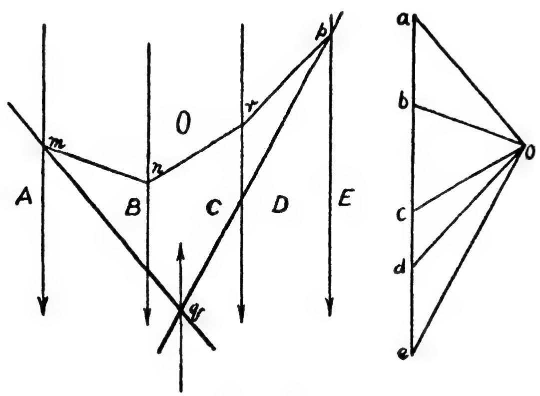


FIG. 290

choose a pole *o* on any side of *ae*. Join *ao*, *bo*, *co*, *do* and *eo*. Take a point *m* in the line of action of the force *AB* in the space diagram and draw *mn*, *i.e.*, *BO* || *bo*, cutting the line of action of the force *BC* at *n*. Next, draw *nr*, *i.e.*, *CO* || *co* to cut the line of action of the force *CD* at *r* and *rp*, *i.e.*, *DO* || *do* to cut the line of action of the force *DE* at *p*. Through *m* and *p* draw *mq*, *i.e.*, *AO* and *pq*, *i.e.*, *EO* parallel respectively to *ao* and *eo*, cutting at *q*. Then, *q* is the point in the space diagram through which the line of action of the equilibrant of the system represented by *ea* will pass.

314. Reactions—horizontal beam with concentrated loads simply supported at two ends. The beam being simply supported at two ends the reactions at the supports will act in the vertical upward direction and they can be fully determined graphically in the following way with the help of force and link polygon. The two reactions and the concentrated loads being in equilibrium, the force diagram and the funicular polygon will be closed figures. Let *MM* be the horizontal beam (Fig. 291) supported at two ends and let four concentrated

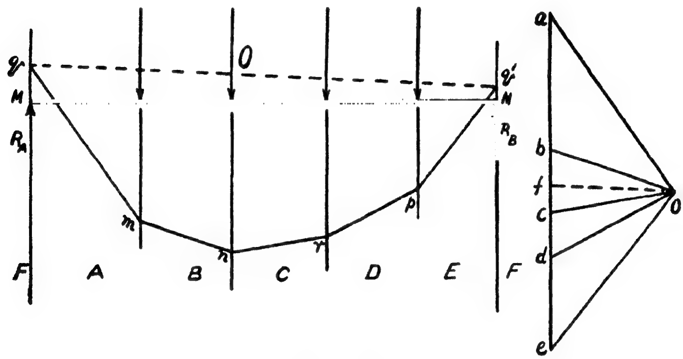


FIG. 291

loads *AB*, *BC*, *CD* and *DE* act on it as shown in the space diagram. Let the reactions be notated by *AF* and *FE*, which can also be named R_A and R_B respectively. Draw the vector diagram as in the previous article and it is clear that the two reactions combining together will be equal to *ea*. Next draw the funicular polygon in the space diagram just in the same way as was done before. Ultimately two points *q* and *q'* are obtained on the lines of action of the reactions *AF* and *FE*. Because the funicular polygon for a system of forces in equilibrium must be a closed one, join *qq'* to close the figure. From the pole *o* in the force diagram draw a straight line *of* parallel to *qq'* Then, *ef* and *fa* give the magnitudes and directions of the two reactions R_B and R_A respectively—R_A and R_B may be said to be the two equilibrants for the four concentrated loads.

315. Each force has been found to be fully defined by three elements—(1) magnitude, (2) direction and (3) the point through which its line of action passes (*i.e.*, position in space). In a system of non-parallel non-concurrent forces in equilibrium if any two of the forces, which may be called two equilibrants of the system, be

not fully defined, then, those two equilibrants may be determined by drawing the force diagram and the funicular polygon side by side. It is to be remembered that this is possible only when any three of the six elements of the two equilibrants are unknown. It is also to be noted here that in case of parallel forces only two unknown elements can be determined.

For example, in the diagrams of the Fig. 288, say, the force *AB*, *BC* and *CD* are fully defined and of the force *DE*, a point *m* is given through which it passes and of the force *EA*, the magnitude and direction are given. To determine the two equilibrants *DE* and *EA* fully. Begin to develop the force diagram and the funicular polygon side by side. Draw the vectors *ab*, *bc* and *cd*. From *a* draw a vector to represent the force *EA* in the opposite direction. Take any point, *o*, in the force diagram. Join *oa*, *ob*, *oc*, *od* and *oe*. Also join *de*, then, *de* will represent the force *DE* in magnitude and direction and thus the force *DE* is fully determined. Now, to draw the funicular polygon, starting from the point *r* in *AB*, *rs*, *st* and *tp* are drawn—the line of action of the force *DE* through *m*, which is parallel to *de*, cuts the straight line *tp* at *p*. Through *p* draw a straight line $pq \parallel eo$ and through *r* draw $rq \parallel oa$ – the two lines cutting at the point *q*. Then, the force *EA* will act through the point *q*. Thus, the two diagrams—(1) force diagram, and (2) funicular polygon—being fully drawn, all the forces are fully determined.

316. Moment of Force. First and Second moments of a force about a point in the plane of action of the force or about an axis perpendicular to the plane.

Let *AB* be the force and *x* be a point in its plane at a distance *d* from its line of action (Fig. 292). It is required to find out the moment of *AB* about *x*, or about an axis passing through *x* and at right angles to the plane of action of *AB*. (Here the plane is the plane of the paper).

First Moment

Analytically, the moment of *AB* about $x = AB.d$. The graphical method of determining the moment is as follows :

In some definite scale draw *ab*, a vector, to represent the force *AB* in magnitude and direction. Choose a pole, *o*, and join *ao* and *bo*. Take a point *o′* on the line of action of *AB* in the space diagram.

From o' draw $O'a'$ and $o'b'$ parallel respectively to oa and ob. Suppose they cut the straight line through x which is parallel to the line of action of AB at a' and b' respectively. Draw oh and

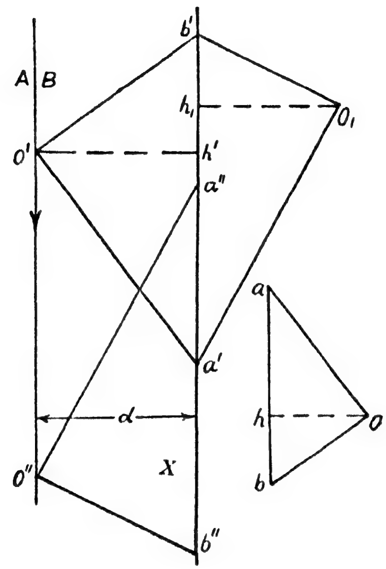


FIG. 292

$o'h'$ perpendiculars from o and o' on ab and $a'b'$ respectively. Then, from similarity of triangles between aob and $a'o'b'$, $\frac{ab}{a'b'} = \frac{oh}{o'h'}$ or, $ab \times o'h' = a'b' \times oh$. But, ab represents the force AB and $o'h'$ is equal to d. Therefore, $ab \times o'h'$, *i.e.*, $a'b' \times oh$ is the measure of the moment of AB about x. If the position of the pole o, which is an arbitrary one, be such that oh is of unit length, then, $a'b'$ will represent the required moment. The length of $a'b'$ multiplied by the moment scale gives the actual value of the moment. The moment scale, in this case, is the product of force-scale and space-scale. If p be the force-scale in drawing the vector diagram and s be the space-scale in drawing the space diagram, then, the moment-scale $= p.s$ units per unit length. But, if oh be taken arbitrarily of any value, then, oh must be either a multiple or sub-multiple of unit length. In the case where $oh = n \times$ unit length, the moment-scale is equal to $n.p.s$ units per unit length.

Illus. Ex. 138. *Take a case where AB has a magnitude of 10 lbs. and d is equal to 3 feet.*

Analytically, the moment $= 10 \times 3 = 30$ ft. lbs.

Now, the graphical method with different choice of scales are discussed below.

I. Force-scale, $1'' = 5$ lbs.

Space-scale, $1'' = 1$ foot.

$ob = 1$ inch.

Then, the vector $ab = 2''$ and $d = 3$ inches and the moment-scale, $1'' = 5 \times 1 = 5$ lb. ft. From the idea of geometry it is very clear that $a' b'$ in the diagram in this case will measure $2 \times 3 = 6$ inches. Therefore, the moment $= 6 \times 5 = 30$ lb. ft.

II. Force-scale, $1''$ = the same.

Space-scale, $1'' = 2$ feet.

ob = the same.

In this case, the vector $ab = 2''$, $d = \frac{3}{2}$ inches and the moment-scale, $1'' = 5 \times 2 = 10$ lb. ft. The measure of $a' b'$ in the diagram drawn for this case will be equal to $\frac{3}{2} \times 2 = 3$ inches. Therefore, the moment $= 3 \times 10 = 30$ lb. ft.

III. Force-scale, $1''$ = the same.

Space-scale, $1''$ = the same as in case I.

$ob = \frac{1}{3}$ inch.

The vector $ab = 2''$, $d = 3''$ and the moment-scale, $1'' = \frac{1}{3} \times 5 \times 1 = \frac{5}{3}$ lb. ft. From the diagram drawn for this case, $a' b'$ will be found to be $2 \times 3 \div \frac{1}{3} = 18''$ long. Therefore, the moment $= 18 \times \frac{5}{3} = 30$ lb. ft.

It is to be marked that the point x may be taken anywhere on the straight line drawn through it parallel to the line of action, and the moment will be the same in all cases. Therefore, this moment about x may be said to be the moment about the straight line through x, which is parallel to the line of action of the force.

It is also to be noted here that $o' a' b'$ can be said to be a portion of the funicular polygon of a system of forces of which AB is one in the space, with respect to the pole, o, in the force diagram.

Second Moment

Take any point o'' on the line of action of AB and draw the funicular diagram, $o'' a'' b''$, in the same way as before with the help of the diagram $o_1 a' b'$. Drop a perpendicular $o_1 h_1$ from o_1 on the vector line $a' b'$. From

similarity of triangles, $o_1 a' b'$ and $o'' a'' b''$, $a' b' \times o' b' = a'' b'' \times o_1 b_1$. But, from the last proof, $a' b' = \frac{ab}{oh} \times o' b'$. Substituting this value in the previous relation,

$$\frac{ab}{oh} \times o' b' \times o' b' = a'' b'' \times o_1 b_1$$

$$\text{or,} \qquad ab \times (o' b')^2 = a'' b'' \times o_1 b_1 \times ob$$

The left-hand portion of the equal sign in the equation represents the second moment of the force. Hence, if $o_1 b_1$ and ob be taken of unit length, then, $a'' b''$ represents the second moment of the force. The length $a'' b''$ multiplied by the second moment scale gives the actual value of the second moment. Second moment scale = Force scale $\times$ (Space scale)^2, when $o_1 b_1$ and ob are of unit length. But, when they are arbitrarily taken, the second moment scale = force scale $\times$ (Space scale)^2 $\times$ $o_1 b_1 \times ob$.

Illus. Ex. 139. *In the Illus. Ex. 138,*

If Force-scale, $1'' = 5$ lbs.,
Space-scale, $1'' = 2$ feet,
$ob = \frac{3}{4}$ inch,
$o_1 b_1 = 2$ inches,

find the second moment of the force about the same axis.

The vector $ab = 2''$, $d = 1\frac{1}{2}$ inches and the moment scale $= 5 \times 2^2 \times \frac{3}{4} \times 2 = 30$ lb. ft^2. If the diagram is drawn with the above data it will be found that $a'' b'' = 3$ inches. Therefore, the second moment of the force $= 3 \times 30 = 90$ lb. ft.^{2}.

The second moment of a force has no physical significance, but the method of determining the second moment of a force is required for finding out the second moment of area, which is an urgent and imperative quantity appearing in strength calculation of machine parts, beams, columns, trusses, etc.

317. First and Second Moments of Plane Areas about an Axis in the Plane of the Areas.

Method 1

Following the principle of the previous article the first and the second moments of the plane areas about an axis in their planes can be easily found out. Regular and irregular figures are divided into small areas and these parts are supposed to denote forces, proportional to these areas, acting through their centres of gravity. The first moment is found out to get the C.G. of the area. Therefore, the question may arise, how to get the points of application of the forces, *i.e.*, the centres of gravity of the different portions of the areas. From the analytical method we know that the C.G. of a

rectangular area lies at the point of intersection of the diagonals. The given area is tried to be divided into smaller rectangular areas as far as practicable.

Illus. Ex. 140. *Find out the first and second moments of the sectional area shown in Fig. 293.*

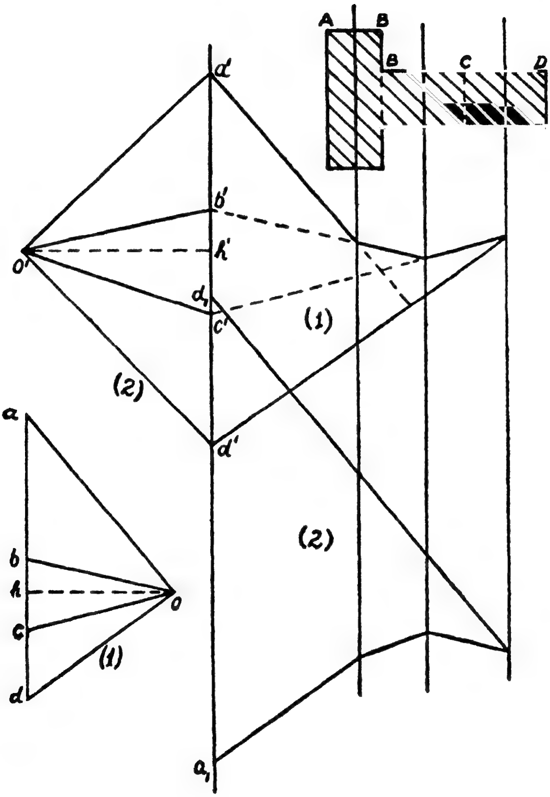


FIG. 293

The area is divided into three smaller rectangular areas as shown. Three forces proportional to the three areas are assumed to act through the centres of gravity of the areas vertically downwards. The lines of action of the three forces will divide the sides *AB*, *BC* and *CD* into two equal parts as shown in the diagram. Draw the vector diagram and the funicular polygon No. 1 as has been drawn in Fig. 293. In the funicular polygon No. 1 *a' d'* represents the first moment of the area about the axis *XX*. Next, draw the vector diagram and

the funicular polygon No. 2. In this funicular polygon $a_1 d_1$ represents the second moment of the area about the same axis. The lengths $a'd'$ and $a_1 d_1$ multiplied by the first and second moment scales respectively give the actual values of the first and second moments of the given area about the axis XX.

If the area scale be $1'' = a$ sq. in.,

space scale, $1'' = d$ inches.

Then, the first moment scale is, $1'' = a.d.ob$ (in)^3

and, the second moment scale, $1'' = a.d.ob.o_1 h_1$ (in)^4

Method 2

There is a general method of determining the first and second moments of irregular areas. The actual diagrams for the areas in proper scale are required to be drawn and the method is described below.

Take the transverse section of a rolled rail section as shown in Fig. 294. To find the first and second moments of the area about

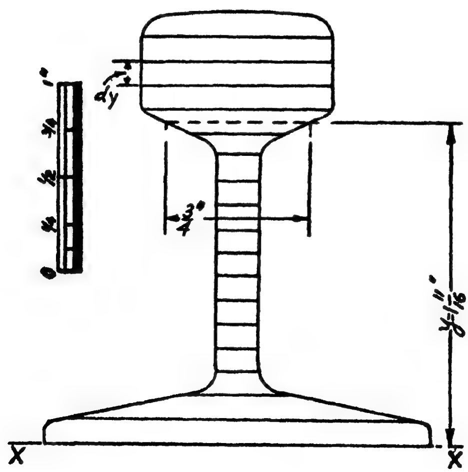


FIG. 294

the axis XX, which coincides with the outer line of the base, also to find the second moment about an axis passing through the C.G. of the area and parallel to the axis XX,

Draw the section to scale and divide the area into a large number of strips parallel to XX and of small thickness dy. Draw the centre lines of these strips. Now, consider each of the strips separately. If the length of the central line of the first strip (extreme end strip) be l_1, then, its area is very approximately equal to $l_1 . dy$, and its moment about $XX = l_1 . dy \times y_1$, where y_1 is the distance of the centre line from the XX axis. Its second moment about the axis is $l_1 \times dy \times y_1^2$. In this way find out the first and second moments of all the strips one by one and add all the similar moments separately. Thus, the first moment can be symbolically represented by $\Sigma\, l . dy . y$ and the second moment by $\Sigma\, l . dy . y^2$.

The position of the C.G. can be determined in the following way :

The first moment of the area about $XX = \Sigma\, l . dy . y$, and the total area $= \Sigma\, l . dy$.

Therefore, the distance of the C.G. from the axis XX,

$$y \quad \frac{\Sigma\, l . dy . y}{\Sigma\, l . dy}$$

Now, finding out the distance of the C.G. from the axis XX, the second moment of the area about an axis passing through the C.G. and parallel to the axis XX is determined as follows :

The second moment of the area about $XX = \Sigma\, l . dy . y^2$. But, we know that $I_x = I_G + A\, d^2$, where I_G is the required second moment, and $d = y$

Hence, $I_G = \Sigma\, l . dy . y^2 - (\Sigma\, l . \mathrm{dy}) . y$

It is evident that the smaller the thickness, dy, is, the greater will be the accuracy in the result.

Illus. Ex. 141. *Find the first and second moments of the area given in Fig. 294 about the axis XX. Height = 2¼ inches.*

The whole height of the area has been divided into 18 equal parts. Therefore, the width of area of each strip, dy, is equal to ⅛ inch. Now, the calculations regarding the first and second moments of the area about XX are shown in the table below.. All the linear measures are multiplied by 32 and put into the table for the advantage of computation.

	y	x	a	ay	ay^2
1.	70	30	4×30	8×1050	16×36750
2.	66	32	4×32	8×1056	16×34848
3.	62	32	4×32	8×992	16×30752
4.	58	32	4×32	8×928	16×26912
5.	54	32	4×32	8×648	16×17496
6.	50	9	4×9	8×225	16×5625
7.	46	7	4×7	8×161	16×3703
8.	42	7	4×7	8×147	16×3087
9.	38	7	4×7	8×133	16×2527
10.	34	7	4×7	8×119	16×2023
11.	30	7	4×7	8×105	16×1575
12.	26	7	4×7	8×91	16×1183
13.	22	7	4×7	8×77	16×847
14.	18	7	4×7	8×63	16×567
15.	14	7	4×7	8×49	16×343
16.	10	9	4×9	8×45	16×225
17.	6	42	4×42	8×126	16×252
18.	2	64	4×64	8×64	16×64
	Total		$4 \times 337 = \Sigma a$	$8 \times 6079 = \Sigma ay$	$16 \times 168779 = \Sigma ay^2$

(I) $y = \dfrac{\Sigma a y}{\Sigma a} = \dfrac{8 \times 6079}{4 \times 337} \div 32 = 1.127$ inches

(II) $I_e = \dfrac{16 \times 168779}{32^4} = 2.576\ (\text{in})^4$

(III) $I_{Ge} = 2.576 - a \times (1.127)^2$

$$= 2.576 - \frac{4 \times 337}{32^2} \times (1.127)^2$$

$$= 2.576 - 1.965 = .611\ (\text{in})^4$$

Method 3

The third method of determining the first and second moments of irregular plane areas is as follows :

Let OX be the given axis and A be the area (Fig. 295). Draw a straight line $O_1 X'$ on the other side of the area parallel to OX, and at a perpendicular distance of S from it. Divide the area A into a large number of strips parallel to OX of small width dy. Let PP be such a strip at a distance, y, from the axis OX. Draw a perpendicular PQ on $O_1 X'$. Choose a pole O' on OX preferably nearest to the area. Join $O_1 Q$ cutting the centre line of the strip PP at P_1. By

similar construction on the other side of the strip a second point P_1 is obtained. Draw a perpendicular from O' on O_1X' cutting the

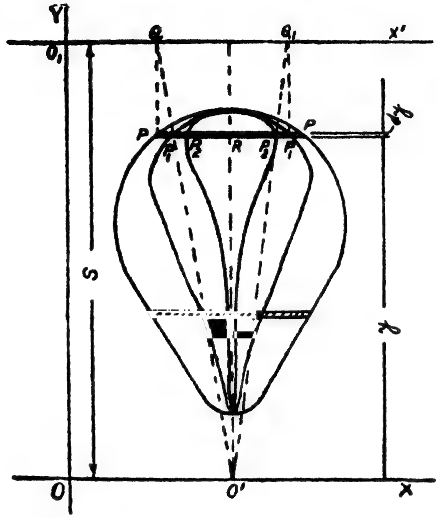


FIG. 295

strip at R. Now, from the similarity of triangles $P P_1 Q$ and $R P_1 O'$,

$$\frac{P P_1}{P_1 R} = \frac{P Q}{RO'} = \frac{P P_1 + P_1 R}{P_1 R} = \frac{P Q + RO'}{R O'} = \frac{S}{y}$$

That is, $\dfrac{P R}{P_1 R} = \dfrac{S}{y}$.

For the other side too the same relation holds good.

Therefore, $\dfrac{P P}{P_1 P_1} = \dfrac{S}{y}$, or, $P_1 P_1 = \dfrac{y}{S} . P P$

Now, the moment of the strip about $OX = P P \times dy \times y$

$$= P P \times \frac{y}{S} \times S \times dy = P_1 P_1 \times dy \times S$$

Treating all the strips in the same way, the moment of the whole area, $A = \int PP \times dy \times y = \int P_1 P_1 \times dy \times S$

$= S \int P_1 P_1 . dy = A_1 . S$, where A_1 represents the new derived area by the aforesaid construction by passing a smooth line through the points similar to P_1 and P_1 obtained from each strip.

A planimeter is required to measure the areas. The C.G. of the area is obtained from the relation,

$$\bar{y} = \frac{\int PP.dy.y}{\int PP.dy} = \frac{A_1}{A}.S, \qquad \text{............}Eq.\ 188$$

where $\bar{y}$ is the distance of the C.G. from OX.

It is to be marked here that the new derived area is proportional to the moment of the area about the axis OX.

From this derived area, following the same method, a second derived area can be obtained as shown in the diagram.

The area of the strip of the second derived area $= P_2 P_2 \times dy$.

But, $P_2 P_2 = \frac{y}{S} \times P_1 P_1$, and $P_1 P_1 = \frac{y}{S} \times PP$

Therefore, $P_2 P_2 \times dy = \frac{y}{S} \times P_1 P_1 \times dy = \frac{y^2}{S^2} \times PP.dy$

or, $\int P_2 P_2.dy = \int P_1 P_1.dy.\frac{y}{S} = \int PP.dy.\frac{y^2}{S^2}$

or, $A_2 = \frac{1}{S^2} \int PP.dy.y^2$, where A_2 is the second derived area.

i.e., $A_2 \times S^2 = \int PP.y^2.dy$, which is the second moment of the area A about OX (I_x).

i.e., $$I_x = A_2 \times S^2 \qquad \text{............}Eq.\ 189$$

Thus, the second derived area is proportional to the second moment of the original area about OX.

Now, $A k^2 = I_x$, where k is the radius of gyration of the area with respect to the axis OX.

Therefore, $A k^2 = A_2 \times S^2$, or, $k = \sqrt{\frac{A_2}{A}} \times S$

The second moment of the area about the axis passing through the C.G. of the area and parallel to OX is from the relation,

$$I_x = I_G + A\,d^2,$$

i.e., $I_G = I_x - A\,d^2$, substituting the values of I_x and d,

$$I_G = A_2 \times S^2 - A\left(\frac{A_1}{A} \times S\right)^2$$

$$= A_2 \times S^2 - \frac{A_1^{\,2}}{A} \times S^2 \qquad \text{............} Eq.\ 190$$

(*b*) A modified method improves the solution in some cases. The method is that instead of selecting a single pole as was done in the previous case different poles are taken for different strips. The poles are chosen at the feet of the perpendiculars from one side of the strips

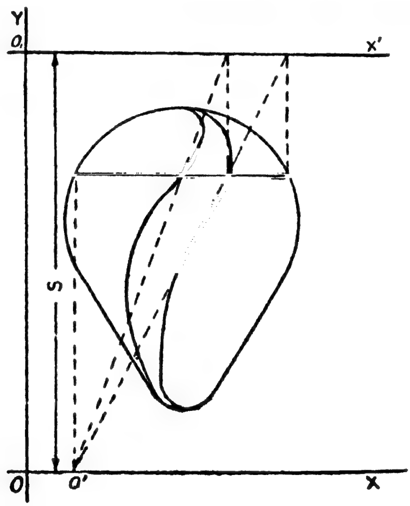


FIG. 296

as shown in the diagram (Fig. 296). The construction is begun from either side, left or right, of the given area. From the construction it is clear that a portion of the boundary line of the area remains common with the boundary lines of the derived areas on the side from which the construction is started. It is clear that the proof remains the same as before. This method is specially suited for the cases where the areas cannot be made symmetrical about an axis vertical to the axis about which the moments are required to be found out.

The diagrams in Figs. 297 and 298 show the different types of diagrams obtained by the previous two methods for a standard rail section.

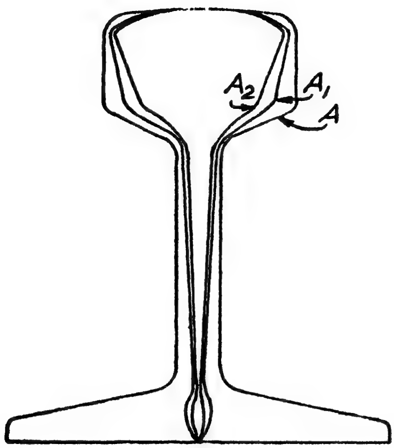


FIG. 297

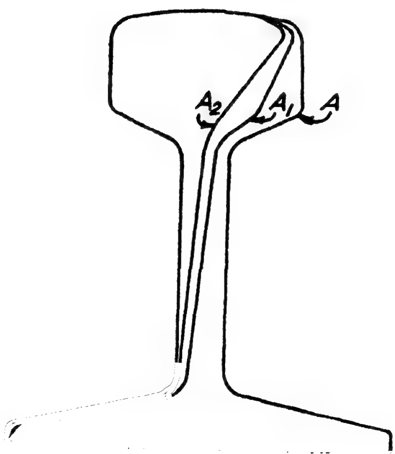


FIG. 298

Method 4

In this method the constructions for the first and second derived areas are the same as before. The only difference is that the actual

area of a section is transposed for advantage, as will be clear from the Illus. Exam. 142, on one side of an axis at right angles to the base line of the area. The method of transposing is explained below. The pole for the derived areas is chosen at the foot of the vertical axis.

*Transposition of area, i.e.,*Transference of area between two rectangular axes.

Draw the given area in the actual scale or in some other definite scale as is advantageous. Produce the base line and draw a vertical line at a distance from the given area. Measure a length on it equal to the altitude of the area. Let the base line be the X-axis about which the first and second moments are to be determined and draw a second axis $O'X'$ at a distance d from the axis OX (Fig. 299). Now, divide the given area into a large number of strips parallel to the base line, preferably of equal widths. Draw the centre lines of these strips and produce them beyond the vertical axis drawn, as shown. Measure along these centre lines from the vertical axis lengths equal to the lengths of the centre lines intercepted by the strip areas (See, the length, $2 \times .825 = 1.65$ inches in the transferred area, Fig. 299). Pass a smooth curve through the tips of these lengths. The area bounded by the vertical axis, the base line and the curve drawn is equal to the given area. It is needless to say that the smaller the widths of the strips are, the greater will be the accuracy in the result.

The first and second moments of the given area about OX.

(a) The first and the second derived areas, A_1 and A_2, are drawn according to the Method 3(b). In drawing the derived areas the strips (in the limit the strips become straight lines) are taken generally at equal distances along the vertical axis of the transposed area as shown in the diagram.

(b) The previous method may be modified in the following way, for the reason that if both the derived areas are developed with the same point O as the pole, the whole diagram becomes most clumsy with too many construction lines within the same space. In order to avoid that and to make the diagram more clear $O'X'$-axis is taken as the reference axis for the second derived area. Along the Y-axis, produced beyond the $O'X'$-axis, equal number of divisions are marked, each of which is equal in length to the corresponding division on the vertical

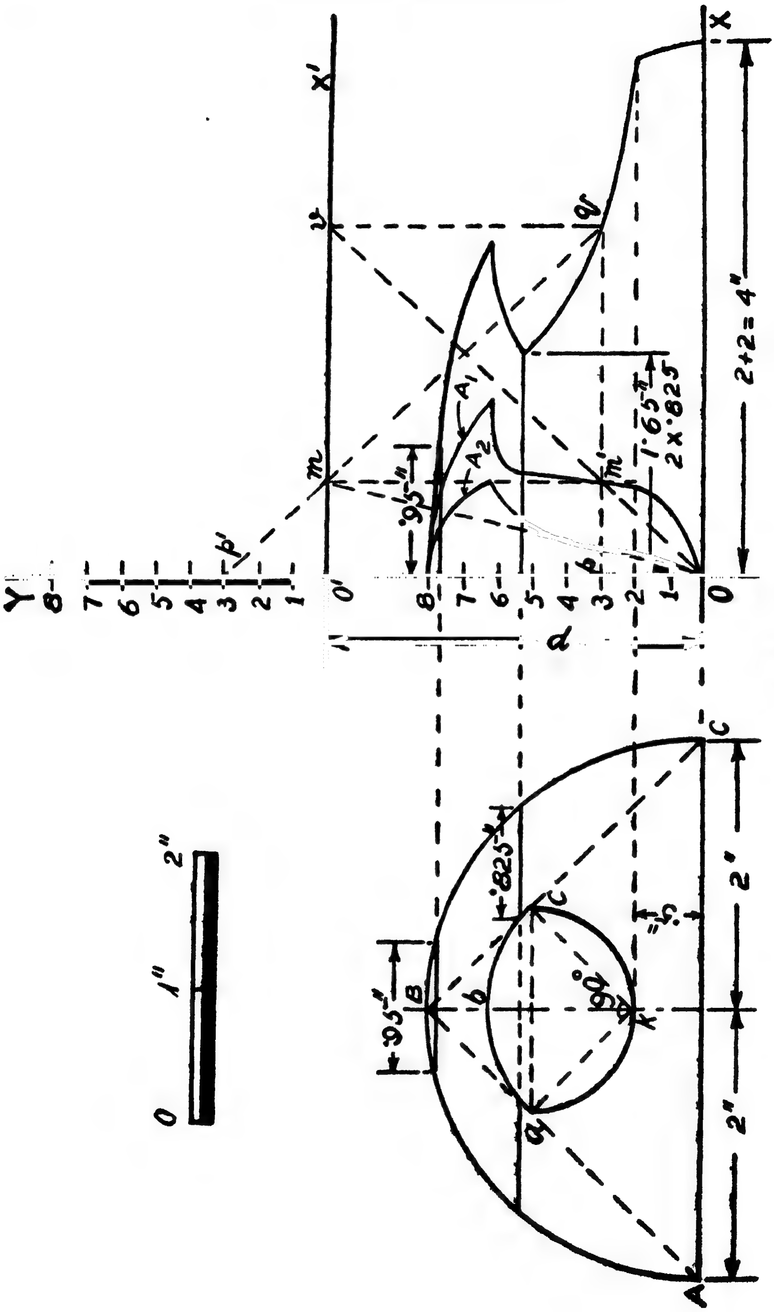


FIG. 299

axis of the transposed area, as shown in the diagram. From the construction it is clear that the strip pq of the transposed area becomes pm' for the first derived area by the method adopted to draw the derived areas.

In order to get the second derived area $m'm$ is drawn perpendicular to $O'X'$ and O and m are joined. Thus, the point for the second derived area corresponding to the point p or m' is obtained. From the construction lines in the diagram it is clear that the point m can also be obtained by joining q with p'—the straight line cuts the $O'X'$-axis at the point m, which is vertically above the point m'. Mark that the strip pq is the third strip chosen and the point p' with which q has been joined is also the third point in the division line $O'Y$. In this way, for each strip, a point like m is obtained on $O'X'$, and joining such points with O, we can get the points for developing the second derived area.

In actual drawing we can avoid, as is obvious from the diagram, drawing the portions of the construction lines, mq and mm', for each strip.

Illus. Ex. 142. *Determine the C.G. of the area as given in the diagram of the Fig. 299 with respect to the X-axis (base line). Also find out the moment of inertia about the same axis. What is the moment of inertia about an axis passing through the C.G. of the area and parallel to X-axis?*

With the help of a planimeter, the given area A and the derived areas, A_1 and A_2, are measured. It is found that,

$$A = 5.1 \text{ sq. inches}$$
$$A_1 = 1.48 \text{ sq. inches}$$
$$A_2 = .7 \text{ sq. inch}$$

The distance between the two axes OX and $O'X'$, d is taken as 2.75 ins.

(1) The distance of the C.G. of the area from the base, *i.e.*,

$$\text{X-axis, } \bar{y} = \frac{A_1}{A} \times d = \frac{1.48}{5.1} \times 2.75 = .8 \text{ inch.}$$

(2) The moment of inertia of the area about the X-axis,

$$I_x = A_2 \times d^2 = .7 \times (2.75)^2 = 5.3 \quad (\text{in})^4$$

(3) The moment of inertia of the area about an axis passing through the C.G. of the area and parallel to the X-axis,

$$I_{Gx} = A_2 . d^2 - A \left(\frac{A_1}{A} \times d \right)^2$$
$$= 5.3 - 5.1 \times (.8)^2 = 2.036 \ (\text{in})^4.$$

318. Bending Moment and Shearing Force in case of Ordinary Beams. If a beam be loaded as shown in the diagram (Fig. 300) with concentrated vertical loads of W_1, W_2, W_3 and W_4 simply resting horizontally on two supports at the two ends, the loads will be balanced by two reactions at the two supports R_A and R_B

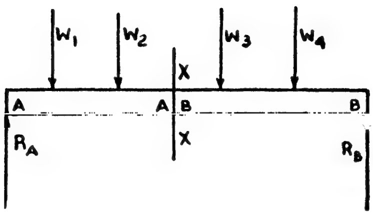


FIG. 300

directing upwards. The beam being in equilibrium under the action of these forces, any portion of the beam may be considered to be in equilibrium under the action of the forces acting on this portion. Divide the beam into two portions, A and B, by an imaginary section XX. Take the portion A. It is in equilibrium under the actions of R_A, W_1, W_2 and the reaction of the portion B on A. The vertical component of this reaction will be such that it is equal to, $R_A - W_1 - W_2$, because under the condition of equilibrium the sum of all the vertical components, *i.e.*, $\Sigma V = 0$.

The reaction of the portion B appears from the property of the material of the beam. The vertical component of this reaction is called the *Shearing Resistance* and the sum of the vertical external forces on the portion A which is equal to the above vertical component of the reaction is called the *Shearing Force* at the section XX.

Again, under the conditions of equilibrium the sum of all the horizontal components acting on the portion A must also be zero. But there is no horizontal component of the external forces acting on A. Therefore, if there be horizontal components due to the reaction of B on A, their vector sum must be equal to zero. If the moments of all the vertical forces acting on A be found out about any point in XX, their sum will produce a clockwise moment when R_A will produce a moment greater than the sum of the moments due to W_1 and W_2. But, under the conditions of equilibrium, the sum of the moments of all the forces about any point in XX must be zero. Therefore, to neutralise the moment due to the vertical components

an anti-clockwise moment must be created by the remaining forces. This anti-clockwise moment must, then, be due to the presence of the horizontal components of the reaction of B on A. The resultant of these horizontal components cannot be a single force, because under the condition of equilibrium the sum of the components must be zero. Hence, the resultant of the horizontal components must form a Couple so that it can produce an anti-clockwise moment of an equal magnitude to the clockwise moment due to the vertical forces. The two equal and opposite resultant horizontal parallel forces, forming the couple—one pulling and the other pushing—appear from within the material of the beam as its property which is called *Stress*. The clockwise moment due to the external vertical forces which has a tendency to bend the beam is called the *Bending Moment* and the anti-clockwise moment due to the horizontal components that appear from the property of the material to neutralise the above bending tendency is called the *Moment of Resistance* at the section XX. Thus, it is found that bending moment is equal to the moment of resistance at any section of the beam. Bending moment is generally denoted by M and shearing force by F.

319. Bending Moment from the Funicular Polygon. (Culmann's Method). The funicular polygon for forces acting on a horizontal beam, $upqrst$, is drawn as shown in the diagram (Fig. 301).

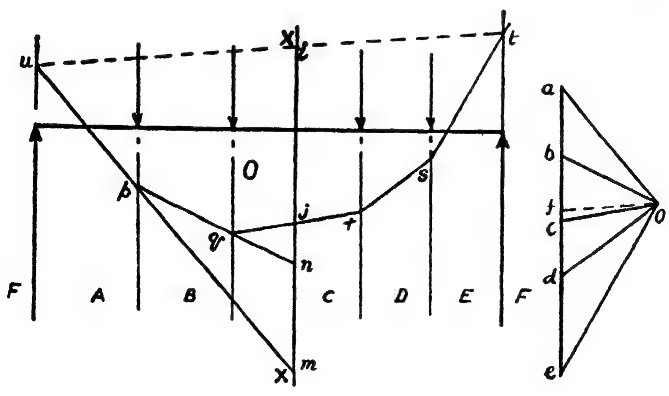


FIG. 301

Produce up ana pq to meet XX at m and n respectively. Then, from the similarity of triangles, uim and ofa it is clear that im represents, in some definite scale, the moment of the force FA about any point in

XX. Similarly, *mn* and *nj* represent the moments in the same scale of the forces *AB* and *BC* respectively about any point in *XX*. Thus, the sum of the moments of the forces *FA*, *AB*, *BC* about any point in *XX* is represented by the length, *im* — *mn* — *nj*, *i.e.*, *ij*, and this, of course, is the bending moment at the section *XX*. The section *XX* being taken anywhere in the length of the beam, it may be said that the depth of the funicular polygon at any section represents the bending moment at that section. A funicular polygon, may, therefore, be taken as the bending moment diagram Bending moment diagram is the diagram from which the bending moment at any section of a beam can be directly read.

320. Shearing Force Diagram drawn from the Force Polygon. Shearing force diagram is the diagram from which the shearing force at any section of a beam can be readily obtained. Draw the force or vector diagram of the forces *AB*, *BC*, *CD*, *DE* and the two reactions, which are determined with the help of the funicular polygon, as was done in the previous case. From the point *f* in the vector diagram (Fig. 302) draw a straight line at right angles to the vector-lines. From this cut off a length to represent the beam length in some definite

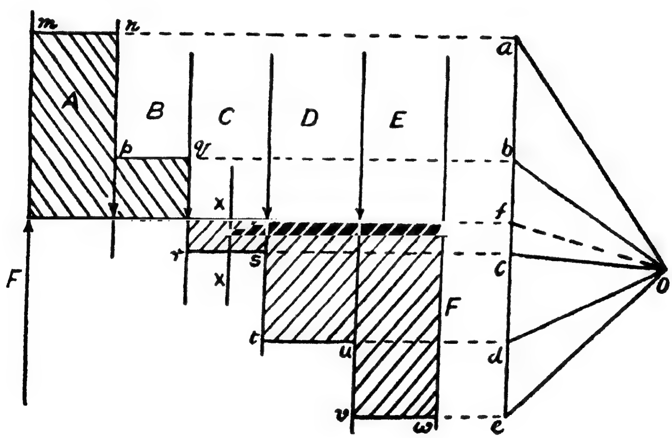


FIG. 302

scale, and in the same scale place the lines of action of the forces in space along the length of the beam. From *a* draw a horizontal line to cut the line of action of the force *FA* and *AB* at *m* and *n* respectively. From *b* draw another horizontal line to cut the lines of action

of the forces AB and BC at p and q, draw a horizontal line from c to cut the lines of action of the forces BC and CD at r and s, from d draw a horizontal line to cut the lines of action of the forces CD and DE at t and u, and from e draw a horizontal line to cut the lines of action of the forces DE and LF at v and w respectively. The area, $mnpqrstuvw$, thus obtained, is shaded as shown in the diagram (Fig. 302). Shearing force at any section is represented by the depth of the area at that section. Here, the shearing force at the section XX is equal to, $FA - AB - BC$, which is represented by, $fa - ab - bc = -fc$; fc is the depth of the shaded area at the section XX. The portion of the diagram above the line representing the beam is taken in the positive sense and below it in the negative sense. It is to be marked that the shearing force at any section between FA and AB is constant; similarly, it is constant between AB and BC, between BC and CD, between CD and DE and between DE and EF.

321. Bending Moment and Shearing Force Diagram for a Cantilever with Concentrated Loads. A cantilever is a beam fixed at one end and free at the other. Whatever may be the nature of the reaction at the fixed end, it is true that its vertical component is equal and opposite to the sum of the external vertical forces, and sum of its horizontal components is equal to zero. The force diagram will be a closed figure and the funicular polygon also will be a closed figure. The two diagrams are drawn as shown (Fig. 303). Mark here that in shearing force diagram the horizontal line representing the length of the beam has been drawn from the point d in the force diagram. Compare this with that of the beam supported at two ends.

In case of a cantilever beam we do not get a complete funicular polygon. A cantilever may be compared with half the portion of a horizontal beam supported at the middle with similar load system on both the sides of the support. For example, take such a beam with two concentrated loads W_1 and W_2 on each side of the support symmetrically placed as shown in the diagram (Fig. 304). The funicular polygon for the loading system will be as shown in the diagram. Now, if we can forget one half of the beam and concentrate our attention to the other half only, then, it will be found that it behaves just similar to a cantilever beam, whose length is equal to half of the beam and which has a loading system just similar to that on one side of the beam. The funicular polygon for this cantilever will be just the same as the portion of the whole funicular

polygon for the loading system of the complete beam symmetrically divided by the line of action of the reaction at the support.

It is to be marked in case of the beam supported at the middle that the reaction at the support is vertical and equal to the sum of the concentrated loads applied. The effect of the resultant clockwise moment of the forces on the right-hand portion about the support is counterbalanced by the effect of the resultant anti-clockwise moment of the forces on the left-hand portion about the same axis.

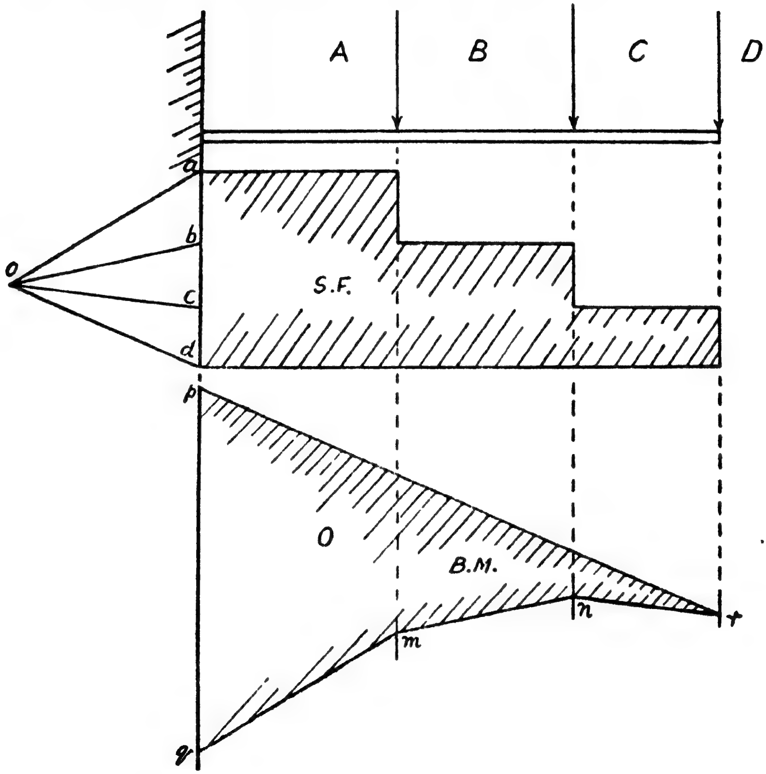


FIG. 303

Therefore, the reaction at the support of a cantilever, to maintain equilibrium, must have a vertical component equal to the sum of the external forces applied and also a moment component to neutralise the

effect of the resultant moment produced by the external loads about the support. The horizontal components at the support produce this moment component.

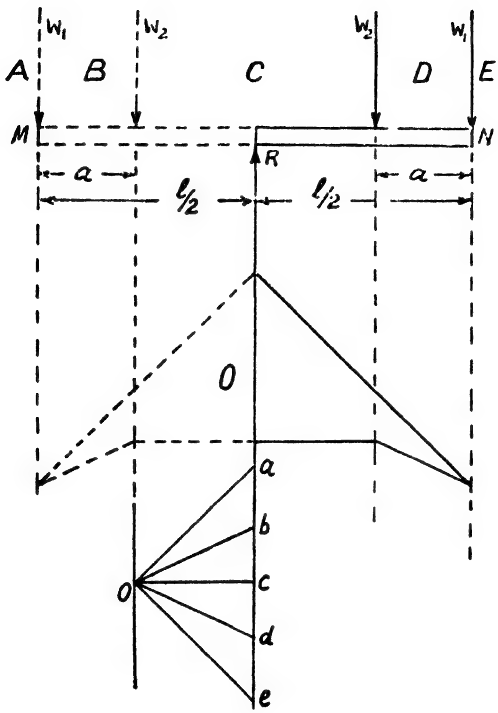


FIG. 304

322. Beams and Cantilevers with Uniformly Distributed Loads. In these cases the length of the member is divided into a large number of equal parts. The load on each part is equal and may be taken to act at the middle of the length. From the funicular polygon of these forces developed from the force diagram the bending moment at any section can be determined as before. The shearing force diagram can also be drawn just in the same way as was done in the previous cases. It is to be noted that the smaller the parts are of the diagram, the more accurately will it give values for bending moment and shearing force at different sections.

323. Stress produced in the Members of Structures.

JOINTED STRUCTURE. The structures are made of a number of bars fastened together by frictionless hinged joints. The whole thing together is called a *Frame* and forms a rigid body. The separate bars are called *Members of the Structures.* Structures are mainly used as roof-trusses, bridge-girders and other frameworks.

A structure is designed in such a way that it can easily withstand the effect of the load that is applied on it. The sections of the members are selected so that the internal resistive force, *i.e.*, the stress produced in the material of each member must be equal to the load borne by it. The whole structure being in equilibrium under the action of load each joint in it, when considered separately, must be taken to be in equilibrium under the action of the load or loads and stresses in the members at the joint. It will be assumed here that, (1) The joint is a point, *i.e.*, the forces keeping the joint in equilibrium meet at a point, and (2) All the forces are co-planer.

It follows, therefore, that the lines of action of the stresses represent the members of the frame-diagram, and the members are subjected either to direct tension or to direct compression. The members in tension are called *Ties* and the members under compression are called *Struts*.

324. Stress Diagram for a Jointed Structure. In designing a structure, therefore, the stress produced in each of the members must be known. A stress diagram or the diagram from which the stresses can be readily read, may be drawn for the whole structure depending upon the condition of equilibrium—the forces acting at each joint must form a closed polygon. Take a simple case as reference. Let three members, A, B and C form a structure (Fig. 305). A load AB is applied vertically at the joint (1) as shown. The magnitudes of the reactions BC and CA are unknown. It is clear that each joint being in equilibrium, the forces acting in it will form a vector triangle. Thus, taking the joints one after another, draw the vector diagrams for all the joints. First, with the given data draw the vector diagram for the joint 1, where one of the forces is completely known and the senses of the other two are known. In drawing the vector triangles for the joints it should be always remembered that the forces must

be taken in the same order, *i.e.*, either in the clockwise or in the anti-clockwise order at each joint. Now, from the vector triangle (1) the magnitudes and the directions of the forces acting through *B* and *A* (with respect to the joint 1) are determined. Next, with the magnitude of the force that is acting through *B* and its sense, the sense and direction of the reaction *BC* and the sense of the force that

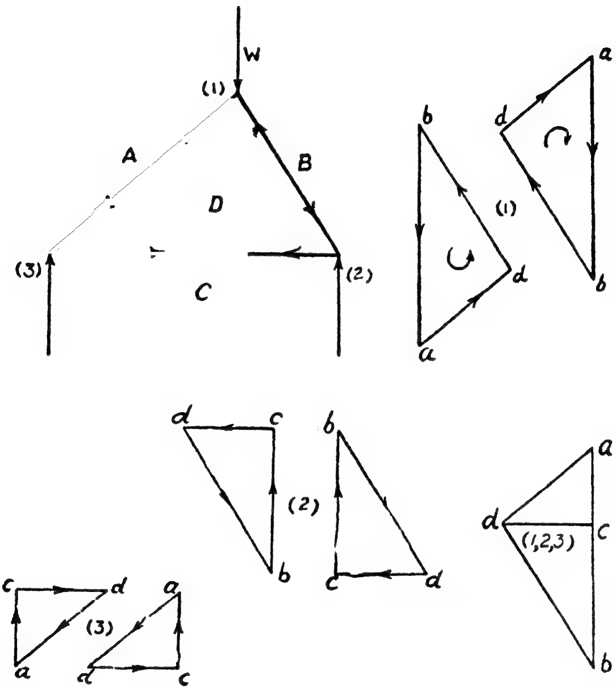


FIG. 305

is acting through *C*, draw the second vector triangle and find the magnitudes of *BC* and *CD* and the directions of *DB* and *CD* with respect to the joint 2. In the same way, with the data obtained draw the vector triangle for the third joint and determine the reaction *CA* and the directions of *DC* and *AD* with respect to the joint 3. Thus, from the directions of the forces acting at each joint it can be easily said whether a member is in tension or compression. It is to be mentioned here that the reactions can also be found out by drawing

the force and funicular polygons for the external force and the two reactions.

Now, combining these three vector triangles a single combined diagram can be drawn as shown. The method of drawing is as follows :

Considering from the joint where the drawing of a complete vector diagram is possible, that is, in this case starting from the joint 1, draw a vector *ab* to represent the force *AB*. Then, taking the forces in clockwise order draw from *b* a straight line *bd* parallel to the member *B* and from *a* draw a straight line *ad* parallel to the member *A*. Thus, the diagram for the joint 1 is complete. Next, in the same diagram go on drawing vectors for the joint 2 keeping similarity with the nomenclature of the forces in Bow's Notation, that is, draw vectors *db*, *bc* and *cd* in order to represent *DB*, *BC* and *CD* respectively. To obtain *bc* and *cd*, draw a straight line *dc* through *d* parallel to *DC*. Thus, the combined diagram is completely drawn. The diagram for the joint 3 may also be checked and it will be found correct.

From the directions of the forces acting through the members *A* and *B* it is found that the members are pushing the joints 1, 2 and 3. Therefore, they are under compression, *i.e.*, they are the struts and the direction of the force acting through the member *C* indicates that it is pulling the joints 2 and 3 and hence, it is in tension and, therefore, the member *C* is a tie member.

Illus. Ex. 143. *Draw the stress diagram for the roof-truss given in Fig. 306 with the vertical loads as shown and determine the force acting through each member. Prepare a table to show the magnitudes of the forces and whether they are in tension or compression.*

The stress diagram is drawn according to scale as shown in the diagram and the table is prepared as required for. The tensile forces will be indicated by the positive signs (+) before the magnitudes and the compressive forces by the negative signs (−).

SCALE :

Space, 1″ = 8 feet

Force, 1″ = 2000 lbs.

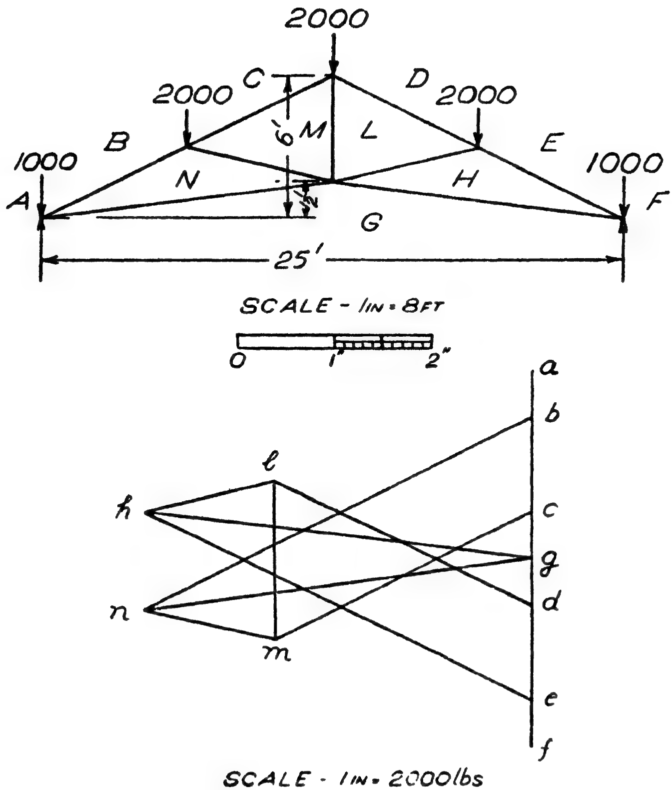


FIG. 306

MEMBERS OF THE TRUSS	ACTUAL MEASURE	MAGNITUDES AND NATURE OF THE FORCES
BN	4.5 ins.	— 9000 lbs
CM	3.0 ins	— 6000 lbs.
DL	3.0 ins	— 6000 lbs
EH	4.5 ins	— 9000 lbs.
HG	4.0 ins.	+ 8000 lbs.
LH	1.38 ins	— 2760 lbs.
LM	1.68 ins	+ 3360 lbs.
MN	1.38 ins	— 2760 lbs.
NG	4.0 ins.	+ 8000 lbs.

N.B. The tension and the compression must be studied from the vector diagram for the balanced system of forces at each joint. It is to be marked whether the force pulls the joint or pushes it. If it pulls, then, the member through which it acts will be in tension and if it pushes, then, the member will undergo compression.

325. Shape of and Tension in Flexible Cords loaded at different Points. If the loads are applied as shown in the diagram (Fig. 307) and if the shape of the cord is as shown, then, when the vector diagrams are drawn for the points of suspension, three triangles are obtained—*oab*, *obc* and *ocd*. If these three triangles are combined in a single diagram, they become the diagram *oabcd*. The vectors

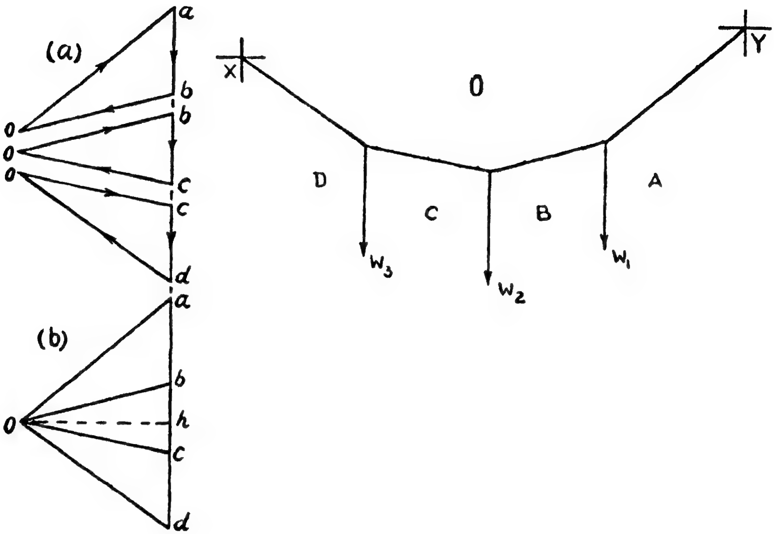


FIG. 307

oa, *ob*, *oc* and *od* which are parallel to the portions of the chain, *OA*, *OB*, *OC* and *OD* respectively, represent the tensions in those portions. From *o* in the combined vector diagram drop a perpendicular *oh* on the line of vectors, *ad*. Then *oh* represents the horizontal component of the tensions in the different portions of the cord or chain and this, of course, is constant throughout the chain. The pull at *X* is the tension *OD* and is the resultant of the vertical component *dh* and the horizontal component *ho*. Similarly, the pull at *Y* is the resultant of the vertical component *oh* and the horizontal component *ha*. In cases of flexible cords if *o* in the vector diagram, which is a fixed point depending on the horizontal component of the tensions in the different portions of the cord, *i.e.*, also the horizontal pull of the reactions at *X* and *Y*, be taken as the pole to draw the funicular polygon in the space diagram, then, the sides of the polygon will coincide with the different portions of the cord if the polygon be

started to be drawn from X or Y. If the horizontal pull be increased or decreased the shape taken up by the cord will alter and the sagging will also be decreased or increased accordingly. The horizontal pull remaining the same, the vertical components of the reactions at X and Y depend on the positions of those two points. The vertical components can be found out with the help of the principle of moments provided the horizontal component be known. Ultimately the horizontal and vertical components being determined, the definite position for the point o is obtained in the vector diagram. Taking this point o as the pole and joining the vector terminii with this point the shape of the sagging cord or chain under the loads can be traced—the sides of the funicular polygon will coincide with the lines representing the positions of the portions of the cord between the loads. It is to be noted here that the weight of the cord has been neglected.

326. In case of transmission line cables where the cross-section and the weight per unit length are constant, it is obvious that the shape of the axis line of the flexible cable will form a regular curve. But, if following the case of a cord with suspended loads as has been done in the previous case—*i.e.*, dividing the length of the cable into n number of equal parts and assuming that the weight of each portion acts through the middle of the length—the force and the funicular polygons be drawn, then, the shape will be as shown in the diagram (Fig. 308). If a smooth curve be drawn which is tangential to the arms of the funicular polygon as shown by the dotted line, then, it will give the approximate shape of the cable axis. It is clear that as n increases the shape in the limit approaches the shape of a smooth curve.

However, if the *dip* or *sag* of the cable be very small, the weight per unit *horizontal distance* between the two supports can be taken as approximately constant. Similar is the case of a suspension bridge as shown in Fig. 309. Neglecting the weight of the suspension cord, chain or cable, which is very small in comparison with the weight of the bridge, we find that the load per unit horizontal distance is constant.

In all such cases, where load per unit horizontal distance may be taken as constant, the curve formed by the axis line of the suspension member is approximately a parabola, which is proved analytically as follows :

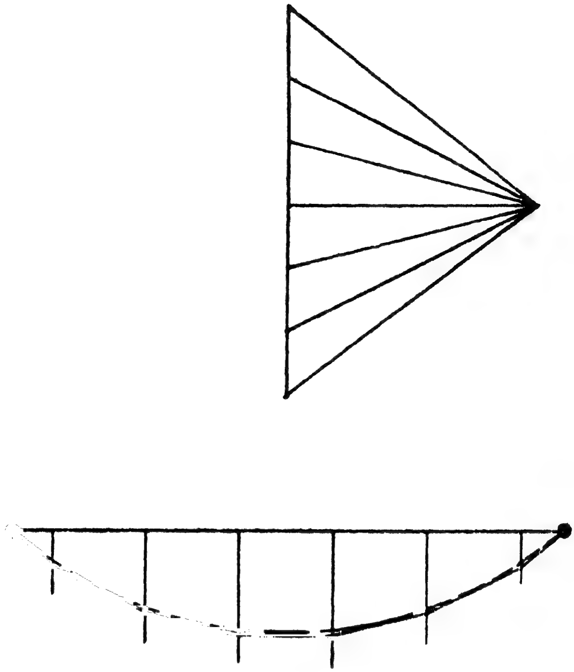


FIG. 308

In Fig. 309 (II), a very small portion of the suspension member has been taken as a free body in equilibrium under the action of the three forces, T, H and wx (let wx be represented by Q).

Then, $H = T \cos \theta$ ... ... (*i*)

$\Sigma V = T \sin \theta - Q = 0$ ... ... (*ii*)

and, $\Sigma M = H . y - \dfrac{Q . x}{2}$

$= H . y - \dfrac{w . x^2}{2} = 0$ ... ... (*iii*)

From (*iii*), $y = \dfrac{w . x^2}{2 H}$ ....*Eq.* 191

which is an equation for a parabola.

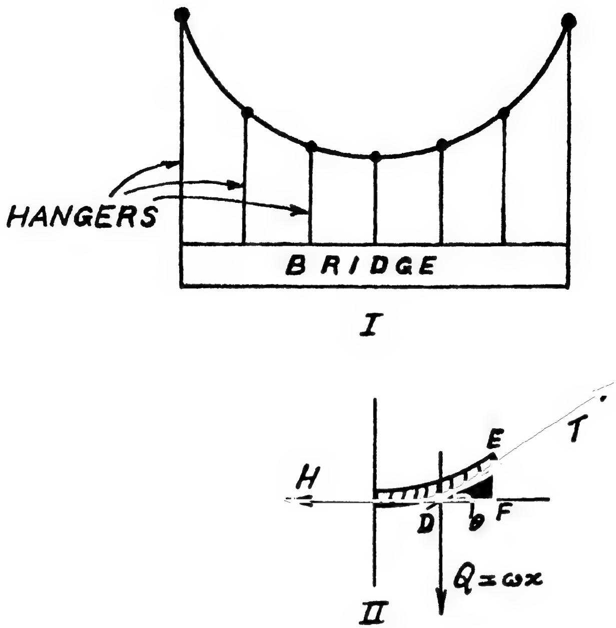


FIG. 309

Again, from the vector diagram *DEF*,

$$T^2 = H^2 + Q^2$$

or, $T = (H^2 + Q^2)^{\frac{1}{2}}$ ....*Eq.* 192

which is the tension in the member at any abscissa x.

Thus, T increases when x increases, and hence, it is maximum at the supports, A and B. That is,

$$T_{max} = \left(H^2 + \frac{w^2 s^2}{4} \right)^{\frac{1}{2}} \qquad \text{............} Eq.\ 193$$

where s is the distance between the two supports, *i.e.*, the span.

From *Eq.* (*iii*), $H = \dfrac{w\, x^2}{2\, y} = \dfrac{w\, s^2}{8\, d}$ ....*Eq.* 194

where d is the dip, *i.e.*, the sag.

The slope of the member at any point E,

$$\tan \theta = \frac{dy}{dx} = \frac{Q}{H} = \frac{w\, x}{H}$$

At the supports the slope is such that,

$$\tan\theta = \frac{w\,s}{2\,H} = \frac{w\,s}{2} \div \frac{w\,s^2}{8\,d}$$

$$= \frac{4\,d}{s} \qquad \cdot Eq.\ 195$$

Note I. In case of a cable if the sag is less than 10% of the span the curve formed by the axis line is approximately a parabola. But, the true curve is a *Catenary* and will be treated afterwards.

Note II. As the bridge is suspended with the help of hangers it is clear that the distributed load of the bridge has been divided into a number of concentrated loads. Therefore, the actual shape of the suspension member will be something like the funicular polygon as shown in Fig. 308.

Length of the parabolic curve.

$$L = 2\int\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} .\ dx, \text{ where } L \text{ is the length.}$$

$$\text{But, } \tan\theta = \frac{dy}{dx} = \frac{wx}{H}$$

$$\text{Therefore, } L = 2\int_0^{\frac{s}{2}}\left[1 + \frac{w^2\,x^2}{H^2}\right]^{\frac{1}{2}} .\ dx = S + \frac{w^2\,s^3}{24\,H^2} - \frac{w^4\,s^5}{640\,H^4} + \cdots\cdots$$

(neglecting the other factors which are very small).

Substituting the value of H from *Eq.* 194,

$$L = S + \frac{8\,d^2}{3\,s} - \frac{32\,d^4}{5\,s^3}. \qquad \cdots\cdots\cdots\cdots Eq.\ 196$$

Illus. Ex. 144. ***A cable of uniform cross-section is suspended from two towers of the same height with a span of 600 feet and a sag of 50 feet. If the weight of the cable is 2 lbs. per unit length (which may be taken as load per unit horizontal distance along the span), find, (a) the tension in the cable at the lowest point, (b) the maximum tension in the cable, (c) the slope of the cable at the support, and (d) the length of the cable.***

(*a*) $$H = \frac{w\,s^2}{8d} = \frac{2.(600)^2}{8 \times 50} = 1800 \text{ lbs.}$$

(*b*) $$T_{max} = \left\{ H^2 + \left(\frac{ws}{2}\right)^2 \right\}^{\frac{1}{2}} = \left\{ 1800^2 + 600^2 \right\}^{\frac{1}{2}} = 1897 \text{ lbs.}$$

(*c*) $$\theta = \tan^{-1} \frac{4\,d}{s} = \tan^{-1} \frac{4 \times 50}{600}$$

$$= \tan^{-1} \tfrac{1}{3} = 18.4^\circ \text{ approximately.}$$

(*d*) $$L = s + \frac{8\,d^2}{3s} - \frac{32\,d^4}{5\,s^3} = 600 + \frac{8 \times 50^2}{3 \times 600} - \frac{32 \times 50^4}{5 \times 600^3}$$

$$= 600 + 11.1 - .1 = 611 \text{ ft. very approximately.}$$

Illus. Ex. 145. *If the maximum allowable horizontal tension in a wire which is supported on two poles of equal height is not to exceed 100 lbs. and if the sag is to be maintained within the limit of 1 inch, find the greatest distance the poles may be placed apart. The weight of the wire is .025 lbs. per foot length.*

$$H = \frac{w\,s^2}{8\,d} \qquad \therefore \quad s = \left(\frac{8\,d\,H}{w}\right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{8 \times 100 \times 1}{.025 \times 12}} = 51.64 \text{ feet.}$$

Illus. Ex. 146. *A chain is suspended from two points A and B in two different levels—the level of B being 40 feet higher than that of A. The horizontal distance between the two points of suspension is 200 feet. The lowest point of the chain is 8 feet below the lower support. The load suspended from the chain, including the weight of the chain, along the span may be taken as uniform as 200 lbs. per foot. Determine the horizontal component of the tension at the support and the two tensions at A and B.*

The curve formed by the axis of the chain will be a parabola following the equation, $y = \dfrac{w.\,x^2}{2\,H}$ (Fig. 310).

Therefore, $$8 = \frac{200 \times a^2}{2\,H} \quad \ldots \quad (i)$$

$$48 = \frac{200 \times (200 - a)^2}{2\,H} \qquad (ii)$$

Dividing (*ii*) by (*i*),

$$6 = \frac{(200 - a)^2}{a^2}$$

or, $5\,a^2 + 400\,a - 200^2 = 0$

From which, $a = 57.8$ feet.

Substituting the value of a in (*i*)

$$H = \frac{200 \times 57.8^2}{2 \times 8} = 41760 \text{ lbs.}$$

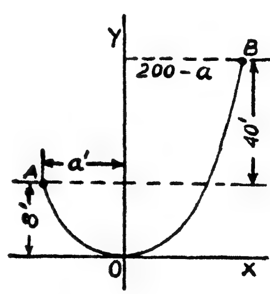


FIG. 310

Load from A to the origin $= 200 \times 57.8 = 11560$ lbs.

Hence, the angle of slope of the chain at A,

$$\theta_1 = \tan^{-1} \frac{11560}{41760}, \text{ from which } \theta_1 = 15.5^\circ$$

Thus, $T_A = 11560 \div \sin 15.5^\circ = 43210$ lbs.

Again, the load from B to the origin $= 200\ (200 - 57.8)$
$= 28440$ lbs.

Hence, the angle of slope of the chain at B,

$$\theta_2 = \tan^{-1} \frac{28440}{41760} = 34.25^\circ$$

Thus, $T_B = 28440 \div \sin 34.25^\circ$
$= 50540$ lbs.

where T_A and T_B are the tensions at the supports A and B.

327. Catenary. Actually in case of a cable where there is invariably sagging due to its own weight in spite of maximum horizontal tension that can be applied before the cable breaks we cannot take that the load on the cable per unit horizontal distance along the span is constant. Therefore, the shape of the axis line of the cable can never be parabolic. It takes a shape which is called a *Catenary*. However, for all practical purposes we can assume the shape as that of a parabola without any appreciable error.

Let a cable take the shape of a catenary as shown in Fig. 311. Take a length AB measured from the vertex equal to l, and also let $H = c.w$, where c is the length of a piece whose weight is equal to the horizontal tension, and w be the weight per unit length of the cable.

Now, $T = \sqrt{H^2 + Q^2}$. Substituting the values of H and Q,

$$T = \sqrt{c^2.w^2 + l^2.w^2} = w\sqrt{c^2 + l^2}$$

Again, $$\frac{dx}{dl} = \cos\theta = \frac{H}{T} = \frac{c}{\sqrt{c^2 + l^2}}$$

Therefore, $$x = c \int \frac{dl}{\sqrt{c^2 + l^2}} = c \sinh^{-1} \frac{l}{c} + c_1$$

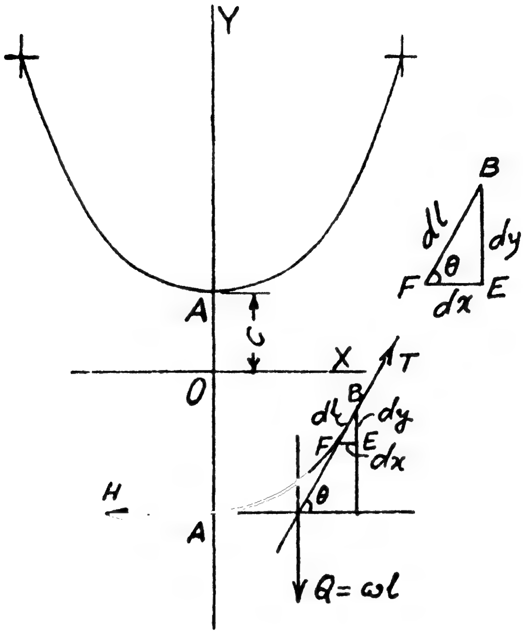


FIG. 311

When $l = 0$, $x = 0$, therefore $c_1 = 0$

Then $x = c \sinh^{-1} \frac{l}{c}$, or, $l = c \sinh \frac{x}{c}$

But $\frac{dy}{dx} = \tan \theta = \frac{w.l}{H} = \frac{l}{c} = \sinh \frac{x}{c}$

Therefore, $y = \int \sinh \frac{x}{c} . dx = c \cosh \frac{x}{c} + c_2$

If the origin is so chosen that it is at a distance of c from the lowest point of the curve, then, the ordinates of the point A are (O, c). Therefore, the constant of integration becomes zero, and

$$y = c \cosh \frac{x}{c} \qquad \cdots\cdots\cdots\cdots Eq.\ 197$$

which is the standard form of the equation of a catenary. The quantity, c, is called the *Parameter* of the catenary.

From the values of x and y,

$$y^2 - l^2 = c^2, \quad \text{or,} \quad y = \sqrt{c^2 + l^2} = c \cosh \frac{x}{c}$$

Therefore, $w.l = w.c \sinh \frac{x}{c} = H \sinh \frac{x}{c}$ ....*Eq.* 198

$$T = w\sqrt{c^2 + l^2} = w.c \cosh \frac{x}{c} = H \cosh \frac{x}{c}$$

$= w.y$ ....*Eq.* 199

If T_1 and T_2 be the tensions at the two ends of a length of a cable, then, $T_1 - T_2 = w\,(y_1 - y_2)$ ....*Eq.* 200

$$L = 2 \int \left[1 + \left(\frac{dy}{dx}\right)^2 \right]^{\frac{1}{2}} .\,dx$$

$$= 2 \int \sqrt{1 + \sinh^2 \frac{x}{c}} \;.\,dx$$

$$= 2 \int \cosh \frac{x}{c} \;.\,dx$$

$= 2\,c \sinh \frac{x}{c}$ ....*Eq.* 201

d (dip or sag) $= y - c$

$= c\left(\cosh \frac{x}{c} - 1\right)$ ....*Eq.* 202

Illus. Ex. 147. *A cable, 200 feet long, weighing 2 lbs. per foot length is stretched between two posts of the same height. The tension at the supports is 300 lbs. Determine the sag, horizontal component of the tension at any section and span.*

I. If we draw the force diagram (vector triangle) for half of the span,

$$\left(w \times \frac{L}{2}\right)^2 + H^2 = T^2,$$

$$\textit{i.e.,}\quad H^2 = 300^2 - (2 \times 100)^2$$

From which, $H = 223.6$ lbs.

II. Parameter, $c, = \frac{H}{w} = \frac{223.6}{2} = 111.8$ feet.

Therefore, according to the *Eq.* 200

$$300 - 223.6 = 2\,(y_1 - y_2)$$

That is, the sag $= (y_1 - y_2) = 38.2$ feet.

III. From the *Eq.* 201

$$100 = 111.8 \times \sinh \frac{x}{111.8}, \text{ or, } \sinh \frac{x}{111.8} = .8945$$

$$\text{Therefore, } \frac{x}{111.8} = .8088, \text{ or, } x = 111.8 \times .8088$$

$$= 90.4 \text{ feet.}$$

Hence the span is equal to $2 \times 90.4 = 180.8$ feet.

Illus. Ex. 148. *A cable is stretched between two points on the same level at a distance of 30 feet. Determine the parameter of the catenary and the length of the curve, if the sag is $\frac{1}{3}$ of the span*

$$y - c = 10, \text{ therefore, } y = 10 + c \quad \dots \quad (i)$$

$$y = c \cosh \frac{x}{c} = c \cosh \frac{15}{c}$$

$$= \frac{c}{2}\left(e^{\frac{15}{c}} + e^{-\frac{15}{c}}\right)$$

$$= \frac{c}{2}\left(1 + \frac{15}{c} + \frac{225}{2c^2} + \frac{3375}{6c^3} + 1 - \frac{15}{c} + \frac{225}{2c^2} - \frac{3375}{6c^3}\right)$$

— expanding the exponential function up to the fourth term.

$$= c + \frac{112.5}{c} \quad (ii)$$

From (i) and (ii),

$$10 + c = c + \frac{112.5}{c}$$

From which, $c = 11.25$ feet

$$\text{Again, } y^2 = c^2 + \frac{L^2}{4}, \quad \text{or,} \quad L^2 = 4\,(y^2 - c^2)$$

$$= 4\,(y + c)\,(y - c)$$

$$= 1300$$

Therefore, $L = 36.06$ feet.

PROBLEMS

379. Find out graphically the resultant of the system of forces given in Fig. 124, *i.e.*, Fig. I (page 241).

380. Determine the equilibrant in magnitude, direction and position in space of the system of forces in Fig. 125, *i.e.*, Fig. II (page 241).

381. Locate the position in space of the equilibrant of the system of forces in Fig. 126, *i.e.*, Fig. III (page 241).

382. What is the perpendicular distance of the line of action of the resultant of the system of forces given in Fig. 127, *i.e.*, Fig. IV (page 241) from the point of application of the 40 lb. force?

383. Determine the first moment and the second moment of a British Standard Beam section as described below, with respect to an axis coinciding with the base line. Also compute the moment of inertia about an axis passing through the centroid of the area and parallel to the base line. If the radius of gyration of the area about an axis at right angles to the previous one and passing through the centroid is 1.717 inches, find the second moment of the area about that axis.

Size	Weight	Rib thickness t_1	Flange thickness at the middle t_2
18″ × 8″	80 lbs per ft.	0.5″	0.95″

Filleting radius, r_1	Rounding radius, r_2	Angle between the flange inner surface and the rib surface
0.77″	0.38″	is 98°

Ans. (Area = 23.53 sq. ins.,) 211.7 $(in)^3$, 3198 $(in)^4$, 1292 $(in)^4$, 69.4 $(in)^4$.

384. Draw the stress diagram for the roof truss (Fig. 105), page 220 and tabulate the forces acting through the members of the truss showing whether they are in tension or compression.

385. Find graphically the compression and tension members in the girder shown in Fig. 106 (page 221). What are the magnitudes of the forces acting through the members?

386. Draw the stress diagram for the girder shown in Fig. 144 (page 248) and find out the members in which big stresses are induced. Measure the magnitudes of the forces producing the stresses.

387. Determine the tension members of the girder in Fig. 145 (page 249). Which of the members sustains the maximum tension and what is the magnitude of the tension?

388. Draw the stress diagram for the roof truss in Fig 146 (page 249) and determine the forces acting through the members due to the effect of loads

389. Fig 148 (page 250) is the diagram for a roof truss with the vertical loads applied at the joints as shown Determine the magnitudes of the forces acting through the vertical members of the truss

390. Fig 149 (page 251) is the diagram for a cantilever girder with the loads applied as shown Draw the stress diagram and determine the forces acting through the members, 1, 2 and 3 Which of them are in tension and which are in compression?

391. Fig 150 (page 251) represents the diagram of one side of the box frame of a tower for an overhead tank with the loads that are expected to act at the joints shown Draw the stress diagram and determine the forces acting through the different members

392. Draw the stress diagram for the crane frame in Fig 137 (page 245) and determine the stress in the diagonal member of the main body portion Is it in tension or compression?

393 Define—Bending moment, Shearing force Moment of resistance, and Resistance to shear Give expression for each of them and explain Where is the bending moment zero in case of a horizontal beam simply supported at ends? Where is the shearing force maximum?

394. A beam 20 feet long is placed horizontally on two supports at the two ends Three concentrated loads of 5, 3 and 2 tons are applied at distances of 8 feet, 13 feet and 18 feet from the left hand support respectively What are the reactions at the two supports? Determine graphically the bending moment and shearing force at a section 5 feet away from the right-hand support Neglect the weight of the beam

Ans 4 25 and 5 75 tons respectively
22 75 ton ft, 3 75 tons

395. Draw the bending moment and shearing force diagram for a beam 17 feet in length placed horizontally on two supports 15 feet apart, one of which is at one extreme end of the beam Two loads of 5 and 3 tons are applied at distances of 5 and 10 feet from the supported end of the beam respectively and a load of 2 tons at the free end of the beam Locate the section where the bending moment is zero What is the shearing force there?

Ans 13 9 feet from the supported end,
5 93 tons (—)

396. Draw the bending moment and the shearing force diagram for a horizontal beam of 15 feet span having uniformly distributed load of 2 tons per foot run of the beam Determine the bending moment and shearing force at 5 and 12 feet from one of the beam *Ans* 50 ton feet, 36 ton-feet,
5 tons (+), 9 tons (—)

397. A cantilever, 10 feet long has two concentrated loads of 2 tons and 3 tons at the free end and at a distance of 5 feet from the fixed end respectively. Determine graphically the bending moment and shearing force at a section 3 feet from the fixed end *Ans.* 20 ton-feet, 5 tons (+).

398. A cantilever, 10 feet long, has distributed load of 2 tons per foot run Draw the bending moment and shearing force diagram and determine the maximum bending moment and shearing force What is the bending moment and shearing force at a section 5 feet from the free end?

Ans 100 ton-feet, 20 tons,
25 ton-feet, 10 tons

399. Draw the bending moment and shearing force diagram for a horizontal beam of 15 feet length supported at two ends There is a uniformly distributed load of 30 tons and a concentrated load of 5 tons at the middle Find the maximum bending moment At what section is the shearing force nil?

Ans 75 ton-feet, at the middle

400. A horizontal beam, 15 feet long, is simply supported at two ends Two concentrated loads of 2 tons each are applied at equal distances of 5 feet from the ends respectively Determine the bending moments at different sections and draw the bending moment curve with Y-axis as the moment axis and X-axis as the length axis

401. Determine the magnitude and direction of the reactions at the supports of the horizontal beam in Fig 133, if it is hinged at A and simply supported at B. Draw the diagram using Bow's notations, and from the diagram read the vertical component of the reaction at A

402. In a suspension-bridge the span is 1000 feet and the sag of the chains is 60 feet If each of the chains carries a load of 1500 lbs per foot horizontally, determine the horizontal and vertical components of the tension at the supports *Ans* 3125000 lbs, 500000 lbs

403. Each chain of a suspension-bridge carries a load of 1200 lbs per foot length of the span The span of the bridge is 400 feet and the sag is 40 feet What is the tension at the lowest point of the chain? Determine the slope angle of the chain at the support Neglect the weight of the chain

Ans 600000 lbs, 21 8°

404 A copper wire is stretched between two poles of equal height If the weight of the wire is 5 lbs. and the sag is 5% of the span, determine the horizontal tension in the wire *Ans.* 12.5 lbs

405. Cables are suspended from two towers of the same height. The horizontal tension with which the cables are stretched is 200 lbs The towers are posted 500 feet apart Determine the sag if the weight of each cable is .25 lb. per foot length of the span. What is the length of each cable?

Ans. 39 feet approximately.
508 feet approximately.

406 The angle of slope of sagging of a cable stretched horizontally is 15°, and the maximum tensile stress induced is 3000 lbs per sq in If the diameter of the cable is 206 in and the curve formed by the axis of the cable is parabolic, determine the distance between the two posts from which the cable is suspended What is the ratio between the sag and the span? The cable material weighs 556 lbs per cu ft *Ans* 201 3 feet, 0335, *i e*, 3 35%

407 A cable, ½ inch in diameter, is suspended from two points at the same level The sag is 10% of the span The tension at the support is 1600 lbs per sq inch If the load per foot along the span is 2 lbs, determine the sag and the span *Ans* 11 65 feet, 116 5 feet

408. A trolly wire, ⅛ sq inch in cross section, has a span of 100 feet and a sag of 1 foot If the wire weighs 40 lbs per 100 feet, find the intensity of tension of the wire at the support *Ans* 4002 4 lbs per sq inch

409. Per foot run of a cable weighs 2 lbs and it is suspended from two points at the same level at a distance of 554 feet If the tension at the support is 1000 lbs and the sag of the cable is 100 feet, find the angle of slope of the cable at the support the tension at the lowest point of the cable and the length of the cable *Ans* 36 8°, 800 lbs, 600 feet

410. A copper wire is stretched between two points at the same level so that the sag becomes 2 feet only If the maximum stress induced in the sections of the wire be 7000 lbs per sq inch, and if copper weigh ,322 lb per cubic inch, determine the distance between the two supports *Ans* 166 5 feet

411. A cable 50 feet long is suspended from two posts of the same height at a distance of 36 feet Determine the sag *Ans* 15 72 feet

CHAPTER XII

SIMPLE MACHINE

328. Machine is the outcome of the attempt, how greater quantity of work can be done by less effort. The contrivance by which manual labour (which includes the time factor) can be saved for doing work is a machine. From the very beginning much development has been done in the line. In every step of our life's journey, directly or indirectly, consciously or unconsciously, we take the help of machines to perform our work. A pair of tongs, a betel-nut cracker, a crowbar, a balance, a wheel barrow, etc., are the instances of machines that are used by us often.

329. A machine is a combination of separate parts, either linked together or rigidly fixed with each other according to the requirement. Generally the machine parts are made of iron. Other materials are also used where required. In a machine force is applied at one part in order to overcome the effect of another force applied at another definite part. The former force is called the *Effort* and the latter is called the *Resistance* or *Load*. It is needless to say that the contrivance should be such that the effort must be less than the resistance, otherwise there is no utility of using a machine. The forces applied may be acting at a point or may be distributed over a surface or a line. The end at which the effort is applied is called the *Effort* or *Driving End* and the other at which the resistance is overcome is called the *Working End*.

330. No machine can move itself nor can it create any motive power. An external force or motive power is applied at the effort end of the machine which creates motions in different parts following a definite law for individual machine and work is done at the resistance or the working end.

331. Mechanical Advantage. For a definite advantage we always use machine. The advantage obtained from a machine is called the *Mechanical Advantage* of the machine. It is measured by the ratio between the load and the effort. If W be the load and P be the

effort, then, $\frac{W}{P}$ is the mechanical advantage, and is represented by the letter M. That is, $M = \frac{W}{P}$ .... *Eq.* 203

332. It is evident that to keep the different parts of a machine in proper places and to regulate their motions there must be guides and bearings. In the guides and bearings frictional forces will act to oppose the motions. Therefore, of the total works done in the machines some portions will be required to overcome the frictional resistances, before we can utilise the machines to perform desired quantities of work. One thing is to be marked that although the effort in almost all cases is much less than the load, the work done at the effort end can never be less than the total work done at the load or working end and the work done against the frictional resistances. Work cannot be obtained out of no work. From this point of view a machine is not at all useful. However, the most favourable reason for using a machine is that the effort required is much less than the load. With less exertion we can work against a big resistance, which is otherwise practically impossible. For example, ordinarily to raise a load of 1 ton exactly a force of 1 ton is required, but with the help of a machine with the application of much less a force we can easily raise the load. We often find that a single man or two are lifting or moving very big masses from one point to another. Now, we get that,

Work done in a machine, *i.e.*, *the input*

= work done by the machine, *i.e.*, *the output*

+ work done against the frictional resistances.

Hence, if P be the effort, W be the load and if there is no frictional resistance (*i.e.*, if we neglect the consideration of the effect of the resistance in the bearings and guides), then, $P.a = W.b$, where a is the displacement of the effort and b is that of the load.

Again, if the effect of the resistance is taken into account and if work done by the resistance is represented by $F.c$, where F is the frictional resistance and c is the distance moved by the machine parts against friction. Thus,

$$P.a = W.b + F.c$$

or, $W.b = P.a - F.c$, which is the quantity of work that

can be obtained out of the machine. This quantity is known as the *useful work* or the output.

333. Efficiency. The ratio, $\frac{\text{output}}{\text{input}}$ is called the *Efficiency* of a machine. From the previous discussions it is clear that this ratio is always less than unity and is a fraction representing the portion of the total work done in the machine, *i.e.*, input. The ratio is generally multiplied by 100 and the efficiency is represented in percentages. Efficiency is denoted by its initial letter E or e.

Velocity Ratio.

The ratio between the distance moved by the effort and the distance moved by the load is the velocity ratio of a machine, and is represented by its initial letter V. Thus, $V = \frac{a}{b}$

$$\text{Hence,} \quad \frac{\text{output}}{\text{input}} = \frac{W.b}{P.a} = \frac{M}{V} = E \qquad \cdots\cdots\cdots\cdots Eq.\ 204$$

In case where $E = 1$, *i.e.*, where frictional resistance is neglected $M = V$.

334. In machines, generally, the ratio between the distances moved by the effort and the load in equal intervals of time is constant, but there is also a deviation from this general rule. The following examples will explain the statement.

Fig. 123 represents a lever and Fig. 312 represents a simple wheel and axle arrangement. In these arrangements it is evident that the

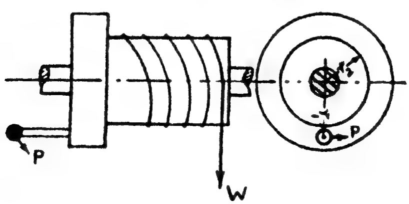


FIG. 312

displacements of the effort and the load are proportional to the lever arms and the diameters of the wheel and axle, in the two cases respectively. Therefore, the displacements of P and W always bear a constant ratio. But, in case of a toggle joint which is shown in

Fig. 313, we find that in the mechanism three links are pin jointed. One is horizontal (as shown in the diagram) through which effort

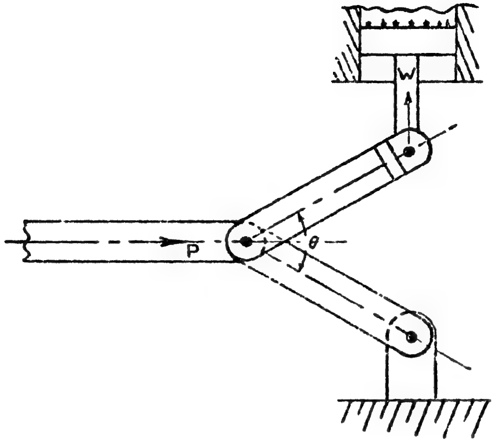


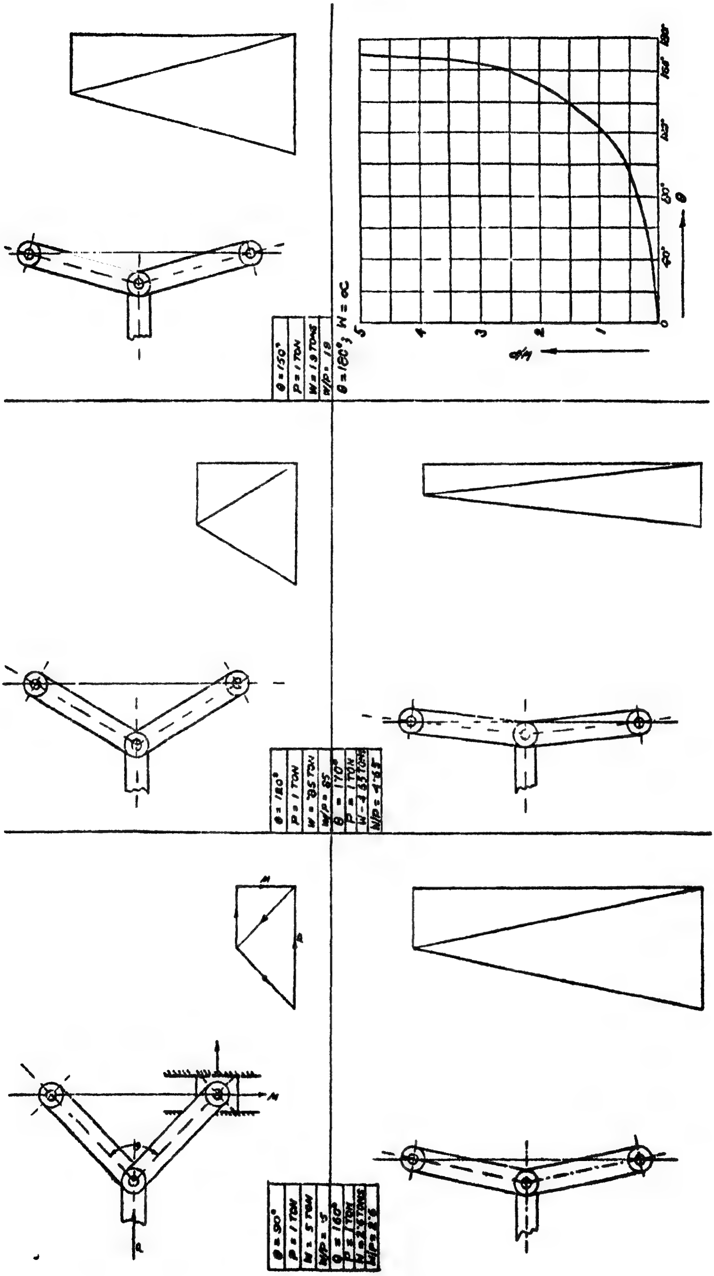
Fig. 313

is applied, the second one is connected with a fixed piece and can have only a motion of rotation about the axis of the joint, and the third one is attached with a piston which is constrained to move in the upward and downward directions within a cylinder. From the arrangement it is clear that the distance moved by the link through which the effort is applied does not bear a constant ratio with the distance moved by the piston within the cylinder which varies with the value of the angle θ between the second and the third links.

In Figs. 314 to 319 in the block on page 482 the relations between the efforts and the loads at different angular positions are shown.

It is to be marked that, as the angle θ approaches 180°, the load approaches *infinity*.

335. Reversed Efficiency. If the machine is reversed, *i.e.*, if W (load) becomes the effort and P (effort) becomes the load, the efficiency under the condition is called the *Reversed Efficiency*. The work done to overcome the frictional resistances only in a machine is equal to $(P.a - W.b)$. This quantity of work is generally taken to remain constant during the working of a machine. Now, so that a



Figs 314-319

machine continues to do its usual function of overcoming a resistance at the new working end even if the machine is reversed, $W'.b$ must be greater than $(P.a - W' B)$. Thus

$$W'.b > P\,a - W'\,b$$

$$\text{or,}\quad 2\,W'\,b > P\,a$$

$$\text{or,}\quad \frac{W.b}{P.a} > \frac{1}{2} \qquad \textit{Eq. } 205$$

Hence, a machine can only be reversed when its efficiency is greater than 50%.

This finding is very useful in designing a lifting machine. During the working of a lifting machine there is always a chance of the machine being reversed if the efficiency of the machine be greater than 50%, to the risk of the life of the workman Therefore, for safety the machine is always tried to be designed so that its efficiency is not greater than 50% For further safety, of course, some additional safety device is adjusted in the machine so that the reversed action, if there be any tendency, is checked thereby.

336. Straight Line Law of a Machine. If the effort-load diagram is drawn with respect to two rectangular co ordinate axes—X and Y, —X-axis representing the load axis and Y-axis representing the effort-axis, as shown in Fig 320, it will be a straight line, AB AO repre-

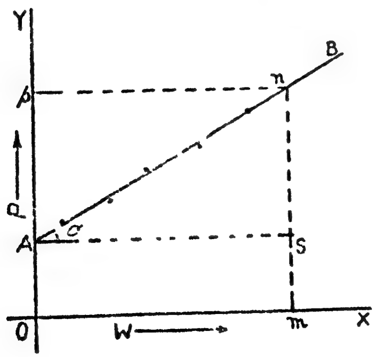


FIG 320

sents the force required to run a machine unloaded. Let this quantity be represented by c.

From the diagram it is clear that the effort bears a constant ratio with the load.. Take a point n on the line AB and drop a perpendicular nm on OX cutting a horizontal line through A at s. In the triangle Ans, $\frac{ns}{As} = \tan\alpha$. It is to be noted that n is an arbitrary point on the line AB and the angle α is constant. Therefore, at any point n on the line AB, $nm = ms + ns = ms + As \tan\alpha$,

$$i.e.,\ P = k\,W + c \qquad \cdots\cdots\cdots\cdots Eq.\ 206$$

where k is a constant and equal to $\tan\alpha$.

It is an equation of a straight line. Now, as the usual relation between the effort and the load in a machine can be represented by a straight line, it is named the *Straight Line Law* of a machine.

337. Maximum Efficiency and Maximum Velocity Ratio of a Machine. If we assume that a machine is frictionless and if P' be the effort required against a load, W, then,

$$P'.a = W.b, \quad \text{or,} \quad P' = W.\frac{b}{a} = \frac{W}{V} \qquad \cdots\cdots\cdots\cdots Eq.\ 207$$

It is to be marked that P' is nothing but $(P - F)$, F being the frictional resistance and P the actual effort. However, if we now consider the machine with its resistance, and if P be the effort required to run the machine, then, the efficiency of the machine,

$$E = \frac{W\,b}{P\,a} = \frac{W}{V\,P}\ . \quad \text{Again,} \quad W.b = P'.a$$

$$\text{Therefore, } E = \frac{P'.a}{P.a} = \frac{P'}{P}$$

$$\text{That is, } E = \frac{\text{effort without friction}}{\text{effort with friction}} \qquad \cdots\cdots\cdots\cdots Eq.\ 208$$

$$\text{Now, } P' = \frac{W}{V}\ , \quad \text{and} \quad P = k\,W + c\ .$$

$$\text{Therefore, } E = \frac{W}{V(kW + c)} = \frac{1}{kV + \frac{cV}{W}}\ ,$$ when W is sufficiently big, $\frac{c\,V}{W}$ can be neglected and the limiting value of the

efficiency becomes $\frac{1}{k\,V}$. Thus the equation for maximum efficiency

$$E_{max} = \frac{1}{k\,V} \quad \cdots\cdots\cdots\cdots Eq.\ 209$$

Again, the mechanical advantage of the machine,

$$M = \frac{W}{P} = \frac{W}{k\,W + c} = \frac{1}{k + \frac{c}{W}},$$

for the same reason as before, $\frac{c}{W}$ can be neglected and the equation for the maximum mechanical advantage becomes,

$$M_{max} = \frac{1}{k} \quad \cdots\cdots\cdots\cdots Eq.\ 210$$

Illus. Ex. 149. *The law of a particular machine is represented by the equation, $P = .05\,W + 5$. Determine the mechanical advantage of the machine when the load $W = 400$ lbs. If the velocity ratio of the machine be 160, find the efficiency of the machine at that load. What is the limiting mechanical advantage and efficiency of the machine? Does the frictional resistance in the machine remain constant? Compute the frictional resistance in the machine when the load is 400 lbs. Form an equation establishing the relation between the resistance and the load.*

1. $P = .05 \times 400 + 5 = 25$ lbs.

2. $M = \frac{400}{25} = 16$

3. $M_{max} = \frac{1}{k} = \frac{1}{.05} = 20$

4. $E = \frac{16}{160} = .01$, *i.e.*, 10%

5. $E_{max} = \frac{1}{m \times v} = \frac{1}{.05 \times 160} = .125$, *i.e.*, 12.5%

6. In case when the efficiency is assumed to be cent per cent, that is, when the machine is considered frictionless, $M = V$, or, $\frac{W}{P'} = V$, or,

   $P' = \frac{W}{V}$, where P' is the effort required if the machine is taken as frictionless. Therefore, $P' = \frac{400}{160} = 2.5$ lbs.

   Hence, the friction at 400 lbs. load, $F = 25 - 2.5 = 22.5$ lbs,

7. Friction, $F = (P - P')$, where P is the actual effort required to move a load. But, $P' = \frac{W}{V}$.

$$\text{Therefore, } F = .05\,W + 5 - \frac{W}{V} = W\left(.05 - \frac{1}{160}\right) + 5$$

$$\textit{i.e.,} \qquad F = \frac{7}{160}\,W + 5$$

SIMPLE MACHINES

338. Simple machines can be classified into three different groups in general,

(1) LEVER, (2) PULLEY and (3) SCREW.

Lever includes Wheel Axle, Toothed Wheel Gear, etc. Screw includes all the mechanism with inclined planes.

339. Lever and Crank. Crank is a piece that can rotate or oscillate about an axis passing through a point at one of its ends. Lever is a machine made up of two crank pieces rigidly fixed having common axis of rotation. The motion of a lever is generally constrained to an oscillation through a small degree. The axis of rotation is called the *Fulcrum* of the lever. The two pieces of the cranks may be in a straight direction or may be made to form any angle whatsoever according to the requirement. According to the formation a lever is named a straight lever (or simply a lever), a bell-crank lever (when the arms form an angle of 90° or less than that) or a rocker (when the angle is greater than 90°).

One end of the lever is called the effort end, where the effort is applied, and the other is called the working end, where the work is done by the machine.

In a straight lever, according to the position of the fulcrum, levers can have three different arrangements. If F be the fulcrum, W be the point of application of the load and P be the point of application of the effort, then, the three arrangements will be as shown below :

F	W	P	W	F	P	W	P	F
.	.	.	.	.	.	.	.	.

Betel-nut cracker, wheel barrow, etc., are machines having lever arrangement of the first type. Balance and generally all kinds of

ordinary levers used as machine parts are of the second type. The third type of arrangement is seen in tongs, forceps, etc.

In a lever if the effort P moves through an angle θ, then, the work done is $P \times \theta \times r_1$ (r_1 being the effort arm). Again, if W be the resistance overcome at the load end, then, the work obtained is, $W \times \theta \times r_2$ (r_2 being the length of the arm at the load end). The

velocity ratio $= \dfrac{\theta\, r_1}{\theta\, r_2} = \dfrac{r_1}{r_2}$ ....*Eq.* 211

340. Wheel-axle Mechanism. It is generally used as a lifting machine (Fig. 312). In this mechanism if an effort P be applied to rotate the wheel, the work done for one revolution is $2\,\pi\, r_1\, P$, and if W be the load lifted, then, the work done by the machine is $2\,\pi\, r_2\, W$. Neglecting the frictional resistances, $2\,\pi\, r_1\, P = 2\,\pi\, r_2\, W$. That is, $P.\, r_1 = W.\, r_2$. Thus, r_1 and r_2 may be taken as the effective lever arms of a lever whose effort arm is r_1 and load arm is r_2. The

velocity ratio, $V = \dfrac{r_1}{r_2}$.

341. Differential Wheel-axle. Figure 321 is a diagram for the differential wheel-axle mechanism. It is used in place of simple wheel-axle arrangement in order to increase the velocity ratio. For one

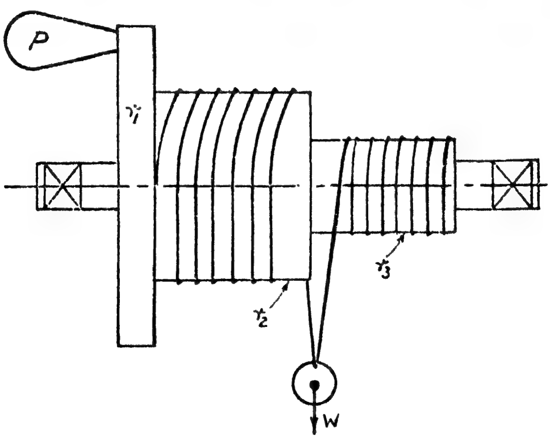


FIG. 321

rotation of the effort P, the axle will also rotate once and in doing so the portion with bigger diameter will draw a length of the rope equal

to $2\pi r_2$, but the portion with smaller diameter will give up a length equal to $2\pi r_3$. Therefore, the shortening of the rope of suspension is $2\pi(r_2 - r_3)$, which allows the load W to be lifted through a distance $2\pi(r_2 - r_3) \div 2 = \pi(r_2 - r_3)$. Thus, the velocity ratio of the machine,

$$V = \frac{2\pi r_1}{\pi(r_2 - r_3)} = \frac{2 r_1}{r_2 - r_3} \qquad \text{............} Eq.\ 212$$

Comparing the relation obtained in a simple wheel-axle it is clear that the velocity ratio in this case is greater, the values of r_1 and r_2 remaining the same in both the cases.

For the same effort the amount of load that can be lifted is greater in this case. Hence, the mechanical advantage, M, which is equal to $\frac{W}{P}$, is also greater.

342. Spur Gear (Toothed Wheel). If the surfaces of two cylinders be in contact and if one of them rotate, then, the other one will also rotate. The force producing the torque = the torque produced by the frictional force between the two surfaces in contact, μR, where μ is the coefficient of friction and R is the normal reaction between the two surfaces in contact. The value of μ can be increased by changing the nature of the surfaces, *i.e.*, making them rough. Now, whatever may be the surface roughness and whatever may be the normal reaction, a slip may creep in during the motion. Therefore, the velocity ratio may differ at any moment and by any amount. The reason and amount of this are not always accountable. To get a definite velocity ratio between the two pieces, teeth are formed on the surfaces of the two so that the question of slip does not arise.

In case of toothed wheels the driving force is not dependent on the normal pressure but is a direct force to produce the necessary torque. Let the two wheels in a gear be represented by their pitch circles (Fig. 322—next page) and also let the pressure angle, *i.e.*, the direction of the application of the force with respect to the common tangent be θ. Draw $O_1 b$, the perpendicular on this line of application of the force from the axis O_1. Then, $O_1 a$ and $O_1 b$ may be taken as the two effective arms of a lever.

Toothed wheels are used in different machines for different purposes. However, the main object for using toothed wheels is to get a definite velocity ratio between two or more rolling bodies.

Take such a pair of wheels—one having 100 teeth and the other 25. Because during the motion each tooth on one meshes with a

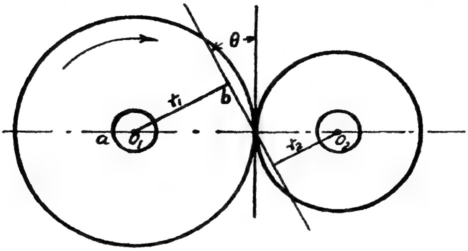


FIG 322

gap of the other, while the first wheel rotates once, the second wheel will rotate $\frac{100}{25}$, *i.e.*, 4 times.

Hence, if the first one be the driver and the second one be the driven, then,

$$\frac{\text{Revolution of the first wheel}}{\text{Revolution of the second wheel}} = \frac{1}{4}, \text{ i.e., } = \frac{25}{100}$$

Hence, $\dfrac{\text{Rev. of the 1st. Wheel}}{\text{Rev. of the 2nd. Wheel}} = \dfrac{\text{Number of teeth in the Driven}}{\text{Number of teeth in the Driver}}$

Eq. 213

343. Toothed Wheels in a Winch. Two kinds of winch are shown in Figures 323 and 324. In Fig. 323 the required velocity

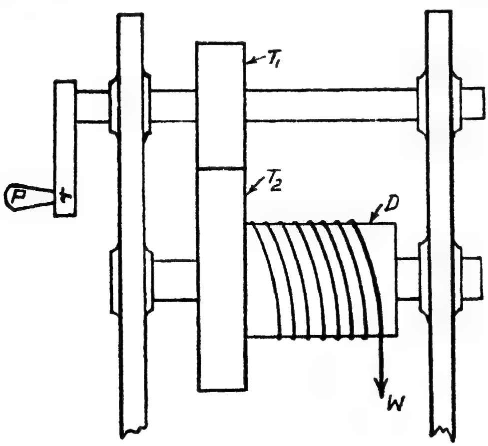


FIG. 323

ratio is obtained in one stage, while in Fig. 324 in two stages. Multiple stages are required to get a greater velocity ratio, *i.e.*, to increase the ratio between the effort and the load. Now, if in the

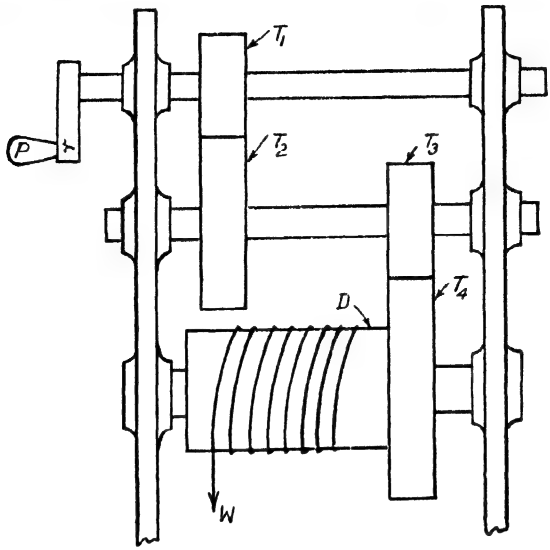


FIG. 324

first case the effort P works for one revolution, the drum D will rotate $\frac{T_1}{T_2}$ times.

$$\text{Therefore, the velocity ratio, } V = \frac{2\pi r}{\frac{T_1}{T_2}\pi D} = \frac{T_2 \cdot d}{T_1 \cdot D}$$

$$\text{Similarly, in the second case, } V = \frac{T_2 \cdot T_4 \cdot d}{T_1 \cdot T_3 \cdot D} \quad \cdots\cdots\cdots Eq.\ 214$$

T represents the number of teeth in the wheel as numbered by the suffix.

In the equations of the velocity ratio in the above two cases d is the diameter of the circle produced by the path of motion of the effort for one revolution and D represents the diameter of the drum. Therefore, the ratio $\frac{d}{D}$ in the equation may be put as $\frac{r}{R}$, where r is the effort arm and R is the drum radius.

Efficiency in a Winch In a winch where the velocity ratio is obtained in one stage only, the efficiency,

$$E = M \times \frac{T_1}{T_2} \times \frac{R}{r}, \qquad Eq\ 215$$

where T_1 and T_2 are the numbers of teeth in the first and second pinions respectively, and r and R are the effort arm and the drum radius respectively.

In a winch where the velocity ratio is obtained in two stages, the efficiency will be as explained below

If E_1 be the efficiency in the first stage, then,
$E_1 \times \text{Input} = (\text{Output})_1$, which means the output in the first stage.

Now, if E_2 be the efficiency in the second stage, then,
$E_2 \times$ Input in the second stage $=$ Output (Final)
That is, $E_2 \times (\text{Output})_1 = \text{Output (Final)}$
or, $E_2 \times E_1 \times \text{Input} = \text{Output (Final)}$
or, $\dfrac{\text{Output (Final)}}{\text{Input}} = E_1 \times E_2 = E$, where E is the efficiency of the machine

$$E = M \times \frac{T_1 \times T_3}{T_2 \times T_4} \times \frac{R}{r} \qquad Eq\ 216$$

Thus, the efficiency, E, is equal to the product of the efficiencies in the two separate stages

In the same way, it can be proved and generalised that in a machine, which is a combination of several mechanisms, the efficiency of the machine is equal to the *product of the efficiencies in the separate mechanisms of the machine.*

PULLEY

344. Pulleys used for hoisting purposes. It is clear that if a single pulley be used to raise a body, the force, *i.e.*, the effort required must be equal to the load lifted There is no advantage in using such a system of arrangement Therefore, in order to get an advantage for hoisting purposes combination of several pulleys with different arrangements are used The pulleys are free to rotate about their

own axles. According to the arrangement there are three systems of combination of pulleys that are used for hoisting purposes, which are described below :

First system : (Fig. 325).

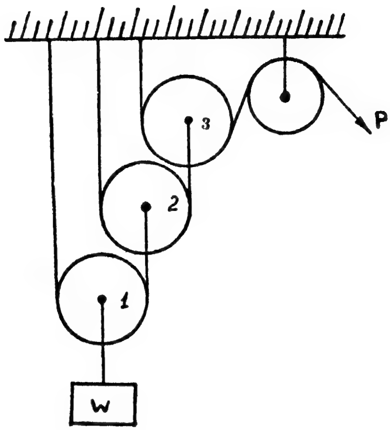


FIG. 325

All the pulleys are movable excepting the last one, which acts as a guide pulley. It is an idle pulley and has no function in the velocity ratio. By the term 'movable' is meant that the pulleys can not only rotate about their own axles but also move vertically up or down. If the load W moves one unit of length upwards, the pulley No. 1 will have to move 1 unit upwards, and therefore, the pulley No. 2 will have to move 2, *i.e.*, 2^1 units in the same direction to keep the cord tight. Similarly, the third pulley will have to move $(2 + 2)$, *i.e.*, 4, *i.e.*, 2^2 units and so on. In case of three pulleys only in a system the effort P will, then, have to move through a distance equal to $(4 + 4)$, *i.e.*, 8, *i.e.*, 2^3 units. Thus, if there are n number of movable pulleys the displacement of P is 2^n units.

By the principle of work, $W \times 1 = P \times 2^n$.

That is, $W = P \times 2^n$

Therefore, the mechanical advantage,

$$M = \frac{W}{D} = 2^n \qquad \cdots\cdots\cdots\cdots Eq.\ 217$$

Second system : (Fig. 326).

There are two sets of pulleys. The pulleys of each set may be on the same axle or may be on different axles as shown. The upper set is fixed, that is, not movable, and the lower set is movable. The axles of each set are rigidly fixed with each other. Proceeding in the same way as was done in the case of the First system it is clear from the diagram that in this case the mechanical advantage, $M = 2n$, ······*Eq.* 218
where n is the number of pulleys in the movable set.

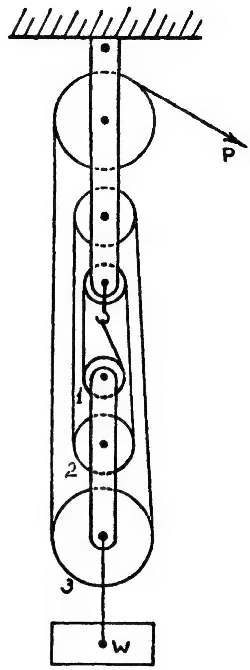


FIG. 326

Third system : (Fig. 327)

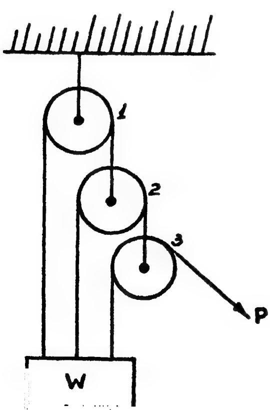


FIG. 327

It is practically the reverse of the first system. The guide or idle pulley is not required in this system and the first one is not movable. In this system, proceeding in the same way as in the previous two, it is found that the mechanical advantage, $M = \dfrac{W}{P} = 2^n - 1$, ············*Eq.* 219

where n is the number of pulleys in the system (not only the number of movable pulleys as in the first and second systems but the total number).

The second system is the most important system and is widely used by the engineers for hoisting purposes. An instance of the third system is found in ships where this system of pulleys is used to hold the masts fixed. Smaller effort can work against a great load. The difference between the first system and the third system with respect to the mechanical advantage is negligible, which is evident from the forms of the equations, but the advantages are that the number of pulleys is less by one and the points of application of the forces on the mass are many, which is a necessary fact or to be considered in many cases.

345. *The second system of pulley is used for hoisting purposes with different arrangements according to requirements which are described below* :

1. Two single sheave blocks—one is fixed and the other movable (Fig. 328).

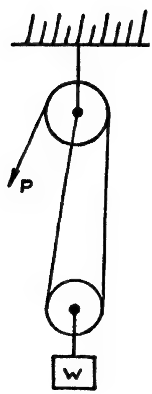


FIG. 328

2. One single sheave block and one double. The upper one is double and the two pulleys are independent of each other

having common axis of rotation, but are not movable (Fig. 329).

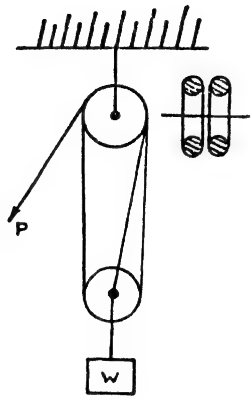


FIG. 329

3. Luff and Luff arrangement (Fig. 330).

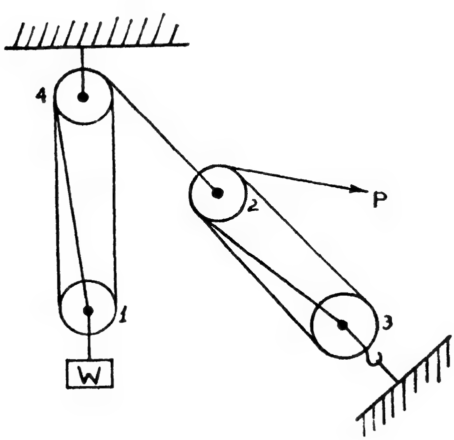


FIG. 330

4. Spanish Burton system (Fig. 331).

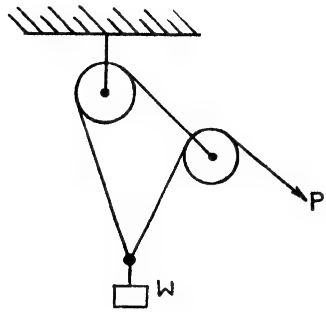


FIG. 331

5. Weston Differential pulley block (Fig. 332).

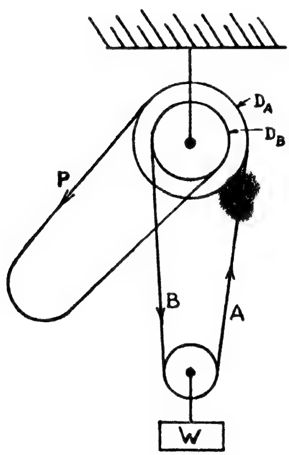


FIG. 332

Neglecting friction

I. In the first arrangement mentioned above,

$$M = \frac{W}{P} = 2, \qquad \text{(ropes about the movable pulley are 2 in number)}$$
$$= V$$

II. In the second arrangement,

$M = 3,$ (number of ropes about the movable pulley is 3)
$= V$

III. In the third arrangement, it is found that it is a combination of two second systems,

$M = \frac{W}{P} = 12,$ Movable pulley No. 1 has 3 ropes and
$= V$ No. 2 has 4, therefore, $M = 3 \times 4 = 12$

IV. Spanish Burton.

Comparing the displacement of P and W it is clear that,

$M = \frac{W}{P} = 3,$ number of ropes attached to the movable
$= V$ pulley is 3. It is a modified form of the 1st arrangement to increase the value of M

V. Weston Differential Pulley Block

Similar to the second arrangement, the only difference is that the upper two sheaves have different diameters and are rigidly fixed with each other. The system is driven by an endless chain. The sheaves have sprockets to fit with the chain links.

Let D_a and D_b be the respective pitch diameters of the upper sheaves which are not movable Also, let D_a be greater than D_b. Due to the application of the effort P in the direction shown by the arrow-head, the upper sheaves will rotate together anti-clockwise. D_a will draw more chain than D_b leaves. For one rotation, the bigger pulley will draw a length equal to $\pi . D_a$ and the smaller one will leave $\pi . D_b$. Therefore, the chain is shortened by $\pi (D_a - D_b)$ for each rotation. This shortening will be equally divided between the two chain portions A and B, on the two sides of the movable pulley.

Hence, W will rise by an amount $= \frac{\pi (D_a - D_b)}{2}$

Therefore, the velocity ratio $= V = \frac{a}{b} = \pi . D_a \div \frac{\pi (D_a - D_b)}{2}$

$$= \frac{2 D_a}{D_a - D_b} \quad \ldots \quad Eq.\ 220$$

If the bearing resistances are neglected, then, we know that the mechanical advantage is equal to the velocity ratio. Therefore, in that case, $M = V = \dfrac{2\,D_a}{D_a - D_b}$.

In case where the bearing resistances are taken into account and if E be the efficiency of the machine, $M \div E = \dfrac{2\,D_a}{D_a - D_b}$.

Illus. Ex. 150. *If in the pulley system in Fig. 329 the load W weighs 1200 lbs., and 3 men weighing 10 stones each stands on W and exerts a pull to sustain the load, determine, (1) the amount of the pull, (2) the tension in the suspending rope of the whole machine. Also determine the pull on the effort end when the men exert it standing on the ground.*

I. $P = \dfrac{1200 + (14 \times 10) \times 3 - P}{3}$

or, $4\,P = 1620 \qquad \therefore \quad P = 405$ lbs.

II. Tension is equal to the total load suspended.

$T = 1200 + 140 \times 3 = 1620$ lbs.

III. The pull will be one-third of the total load suspended, as the velocity ratio is 3.

$P = \dfrac{1200}{3} = 400$ lbs.

Illus. Ex. 151. *The smaller diameter of the upper sheave in a Differential Pulley Block is 13 inches. The velocity ratio of the machine is 15. Find the other diameter of the upper sheave. Taking the machine as cent per cent efficient, determine the amount of load that can be raised by a pull of 50 lbs.*

I. The velocity ratio, $V = \dfrac{2\,D_a}{D_a - D_b}$

Substituting the numerical values,

$$15 = \frac{2\,D_a}{D_a - 13}, \quad \text{or,} \quad D_a = \frac{13 \times 15}{13}$$

$$= 15 \text{ inches.}$$

II. $V = M = \frac{W}{P}$, or, $W = V.P$

Substituting the numerical values,

$$W = 15 \times 50 = 750 \text{ lbs.}$$

SCREW

346. If a helical groove is cut round the surface of a cylindrical rod, leaving metal forming helical strip between the successive turns of the groove, then, the cylinder is said to form a Screw. The helical strip is called the 'Screw Thread'. If it is possible to unwind the whole helical strip retaining its inclination with a transverse section of the rod and to form a straight strip, then, the surface of the thread can be said to form an inclined plane and the motion on this surface will follow the principles of motion on the inclined plane. The angle of inclination as described is the angle of helix formed by the thread.

If a second piece of metal having a cylindrical hole is taken and helical groove is cut at the surface of the hole so that the screw thread of the other piece just fits into the groove of this piece, then, this second piece is called to form a 'Nut' for the other piece.

347. **The Relative Motion between a Nut and a Screw.** The relative motion is just similar to the motion between a slide and an inclined plane, as shown in Fig. 333. The force required to move a body up and down an inclined plane has been described in details in Chapter VIII. From the figure it is clear that when the plane moves to the left the slide rises up, and the rise = the distance through which the plane moves $\times \tan \alpha$.

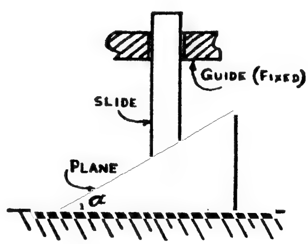


FIG. 333

Similarly, when a nut or a screw is given one rotation, one of the two being kept fixed, the displacement of the moving part is equal to $2 \pi r \tan \alpha$, where r is the mean radius of the sliding part, *i.e.*, of the thread or the groove. The displacement of the moving part along the direction of its axis due to one rotation is called the *Lead* of the screw. If a single helical groove is cut to form a screw and a nut, then, we call the pair a single-threaded screw and nut. If there are multiple

threads, then, the pair is called a multiple-threaded screw and nut. The distance between two successive threads along the direction parallel to the axis is called the *Pitch* of the screw. The lead and the pitch are generally represented by the letters L and p respectively.

Therefore, in a single-threaded screw the pitch is equal to the lead. In case of multiple-threaded screw, $p = \frac{L}{n}$, where L is the lead and n is the number of helical grooves cut.

348. For different purposes different types of threads are used. Generally, for two different purposes the screw-nut mechanism is used.

1. For transmission of motion and power,
2. For holding purposes (for the purpose of rigidly fixing the two bodies together).

For transmission of motion and power generally the following two types are used :

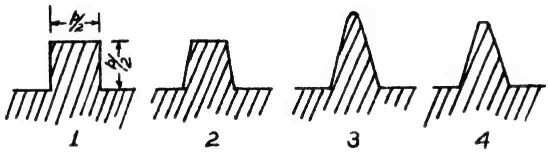


FIG. 334

1. Square-threaded screw and nut (Fig. 334-1).

   Example—Jack and Screw.

2. Acme-threaded screw and nut (Fig. 334-2).

   Example—Lead Screw of a Lathe.

Acme thread is stronger than the square thread as is evident from the shape—the root is thicker than the head. The nut is split into two halves and, thus, the arrangement makes it possible to break and join the connection with the screw by easy push to move the two pieces away and to bring them close together through a small displacement.

3. V-threaded screw and nut (Fig. 334-3 and 4).

It is generally used for holding purposes, such as bolt and nut that are used to rigidly fix one part of a machine with the other. It is some-

times used for the transmission of motion and power in light apparatus. The type 3 represents the thread cut according to the British system and is called *Whitworth Thread* and the type 4 represents the thread cut according to the American system and is called *Seller's Thread.*

The angle between the threads of a Whitworth system is 55° and that between the threads of the Seller's system is 60°. The further difference is that in British system the threads have rounded crest and bottom, whereas, in American system they have flat crest and bottom.

349. According to the direction of helices the screws are called

1. Right-handed
2. Left-handed

The two types are shown in Fig. 335. Mark the inclination of the threads. In the right-handed screw the threads in proceeding from left to right slope downward, and in the left-handed screw they slope upward.

FIG. 335

To know the right-handed and left-handed screws the simple and general method that should be adopted by the beginners is as follows :

Place the nut facing towards you with your left hand and keep it fixed. Introduce the screw with your right hand from the front face of the nut and rotate the screw. If the screw travels away from your body, it is a right-handed screw, otherwise it is left-handed. The common use of right-handed screws is found in fixing the corrugated iron sheets with the roof-truss of a shed. Except for special requirements, generally, right-handed screws are always used.

350. Relation between the Speeds of the Nut and Screw. Ordinarily, the speed ratio for all practical purposes is taken as $\frac{p}{2\pi R}$, where R is the radius of the effort circle.

But actually that is not the case. Take the case of a Jack Screw (Fig. 336). In this case we find that the actual speed ratio is

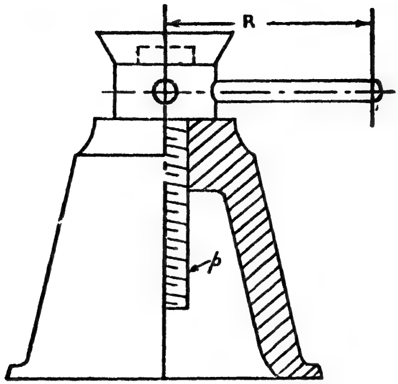


FIG. 336

$\dfrac{p}{2\pi R}\cos\alpha$, where α is the angle of the helix. Because this angle is generally very small, the value of $\cos\alpha$ is taken as 1, without causing any appreciable error.

351. Efficiency of a Screw. The efficiency of a machine being equal to the work obtained from the machine divided by the work done in the machine, in case of screw-nut mechanism, the efficiency,

$$E = \frac{W \times p}{P \times \pi d} = \frac{p}{\pi d} \times \frac{W}{P} = \frac{\tan \alpha}{\tan(\alpha + \lambda)} \quad \cdots\cdots\; Eq.\ 221$$

352. Maximum efficiency of a Screw.

$$E = \frac{\tan \alpha}{\tan(\alpha + \lambda)}$$

$$= \frac{\sin \alpha \cos(\alpha + \lambda)}{\cos \alpha \sin(\alpha + \lambda)} = \frac{\sin(2\alpha + \lambda) - \sin \lambda}{\sin(2\alpha + \lambda) + \sin \lambda}$$

$$= 1 - \frac{2 \sin \lambda}{\sin(2\alpha + \lambda) + \sin \lambda}$$

Now, the efficiency will be maximum, when $\dfrac{dE}{d\alpha} = 0$, and $\dfrac{d^2E}{d\alpha^2}$ is negative.

$$\frac{dE}{d\alpha} = -2 \sin \lambda \frac{-2 \cos (2\alpha + \lambda)}{\{\sin (2\alpha + \lambda) + \sin \lambda\}^2}$$

$$= \frac{4 \cos (2\alpha + \lambda) \sin \lambda}{\{\sin (2\alpha + \lambda) + \sin \lambda\}^2}$$

This can be equal to zero when $\cos (2\alpha + \lambda) = 0$, *i.e.*, when $\cos (2\alpha + \lambda) = \cos \frac{\pi}{2}$, *i.e.*, when $2\alpha + \lambda = \frac{\pi}{2}$, or, $\alpha = \frac{\pi - 2\lambda}{4}$.

Again, $\frac{d^2E}{d\alpha^2} = 4 \sin \lambda . \frac{-2(1 + \sin \lambda)^2}{(1 + \sin \lambda)^4}$, which is a negative quantity.

Hence, the efficiency is maximum when α, the angle of inclination of the thread, is equal to $\frac{\pi - 2\lambda}{4}$.

353. Compound or Differential Screw. The arrangement of the mechanism is shown in Fig. 337. In order to increase the velocity ratio in the screw-nut mechanism this arrangement is done. The screw rod has two portions, S_1 and S_2, of two different diameters. The body N_2 and the load carrier N_1 serve the function of nuts. The body N_2 and the guide G combining together form a composite body. For the rotation of the screw by a handle of effective radius R, the differential screw rod gets a displacement with respect to the body (*i.e.*, nut N_2). If the handle is rotated once in the clockwise direction (seen from the top), the threaded rod will move upward by an amount p_2 as the rod has left-handed thread. For the same reason the load carrier N_1 which is restricted by the guide to move only in the direction parallel to the axis of the rod, will move in the downward direction by an amount p_1. Therefore, the net displacement of the load is $(p_2 - p_1)$ in the upward direction (p_2 being greater than p_1). Thus, the work done by the effort for each rotation remaining constant, the load lifted by the displacement of N_1 can be increased by this differential arrangement. For example, in case of a Jack-screw, by

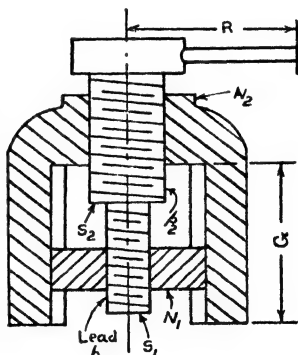


FIG. 337

the principle of work, $2\pi RP = W.p$, neglecting the frictional resistance, but in this case, $2\pi RP = W_1(p_2 - p_1)$. The effort P and the moment arm remaining the same for both the cases, $W_1.p = W_2(p_2 - p_1)$. Now, if p_2 is equal to p and p_1 has a smaller value, $(p_2 - p_1)$ is less than p, and, therefore, W_2 must be greater than W_1.

354. Worm and Wheel. This machine is a combination of a threaded rod and a toothed wheel (Fig. 338), the axle of which bears the load carrier. From the first sight it appears that the machine is a combination of a screw and a lever, but, really it can be taken as a

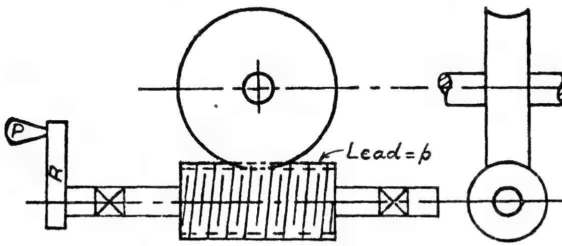


FIG. 338

screw-nut mechanism. The toothed wheel takes the place of a nut and the rod for a screw. The motion of the wheel exactly follows the characteristic of a nut, which can be easily understood from the study of the motion of the rod and the wheel.

The axes of the two pieces, the rod and the wheel, are at right angles to each other. The wheel is 'close fitting', *i.e.*, the element of the wheel surface instead of being a straight line follows the curve of the cylindrical rod on which the threads are cut. The threaded rod is called the *Worm of the Mechanism*. If for one rotation of the worm one tooth of the wheel is pushed off and if T be the number of teeth in the wheel,

$$\frac{\text{rotation of the wheel}}{\text{rotation of the worm}} = \frac{1}{T}$$

Therefore, the velocity ratio,

$$V = \frac{2\pi R}{\frac{1}{T}\pi D},$$ where D is the pitch diameter of the wheel.

$$= \frac{T\,d}{D}, \qquad \cdots\cdots\cdots\cdots Eq.\ 222$$

where d is the diameter of the circle produced by the displacement of the effort.

It is always found that the mechanism can never be reversed, *i.e.*, by rotating the wheel the worm cannot be rotated. Hence, the efficiency of the worm and wheel mechanism is less than 50%. This is a great advantage for using this mechanism in a hoisting machine.

If the thread on the worm forms a single helix, then, the rod is called a *Single-threaded Worm*, and if there is a number of threads forming multiple parallel helices, then, it is called a *Multiple-threaded Worm*, such as, if there are two threads, then, it is called a *Double-threaded Worm*, if there are three threads, it is called a *Triple-threaded Worm*, and so on.

Lead remaining the same, a single-threaded worm pushes one tooth of the wheel for one turn by a linear displacement equal to the lead or pitch of the screw—pitch of the screw and the toothed wheel being the same. A double-threaded worm will push two teeth for the same reason by a total linear displacement of the same quantity, a triple-threaded worm will push three teeth, and so on. Hence, in case of a multiple-threaded worm,

$$\frac{\text{rotation of the wheel}}{\text{rotation of the worm}} = \frac{n}{T},$$

where n is the number of threads on the worm and T is the number of teeth on the wheel.

Therefore, rotation of the wheel $= \frac{n}{T} \times$ rotation of the worm.

Therefore, velocity ratio of the machine,

$$V = \frac{\text{displacement of the effort}}{\text{displacement of the load}} = 2\pi R \div \frac{n}{T} \times \pi D$$

$$= \frac{T}{n} \times \frac{2R}{D} = \frac{T\,d}{n\,D}, \qquad \cdots\cdots\cdots\cdots Eq.\ 223$$

where $d = 2R$.

Illus. Ex. 152. *In a lifting machine Worm and Wheel mechanism is used. The diameter of the drum is 24 inches. The number of teeth in the wheel is 120. Determine the number of threads in the worm in a lead, if one rotation of the effort arm, which is 15 inches long, raises the load by 1.884 inches. If the machine is assumed to be cent per cent efficient, what is the mechanical advantage? What is the mechanical advantage of the machine if the machine is 35% efficient?*

I. The distance through which the load is lifted for one rotation of the effort $= \frac{n}{T} \times$ circumferential length of the drum, where n is the number of threads and T is the number of teeth on the wheel.

Therefore, $1.884 = \frac{n}{120} \times 3.14 \times 24$

or, $n = \frac{1.884 \times 120}{3.14 \times 24} = 3$

II. $M = V = \frac{2\pi R}{1.884} = \frac{2\pi \times 15}{1.884} = 50$

III. $M = .35 \times 50 = 17.5$

Illus. Ex. 153. *Worm and wheel mechanism is used for the purpose of a lifting machine. The worm is triple-threaded and the drum which is rigidly fixed with the wheel having common axis of rotation, has a diameter of 25 inches. Determine the number of teeth on the wheel if 40 turns of the worm move the load up by 30 inches. If the handle attached to the worm has an effective length of 15 inches and if the load lifted weighs 5000 lbs., compute the mechanical advantage of the machine. Take the efficiency of the machine as 40%.*

For 40 turns, $40 \times 3 = 120$ teeth are pushed.

The load rises 30 inches and the diameter of the drum is 25 inches.

Therefore, the angular displacement of the drum $= \frac{30}{\pi \times 25}$

120 teeth pass for $\frac{30}{\pi \times 25}$ of a rotation of the drum or wheel.

Therefore, $\frac{120 \times 25\pi}{30} = 314$ is the number of teeth on the wheel.

Now, $\frac{\text{Angular speed of the wheel}}{\text{Angular speed of the worm}} = \frac{3}{314}$

By the principle of work,

$$.4 \times P \times 2\pi \times \frac{15}{12} = 5000 \times \frac{3}{314} \times 3.14 \times \frac{25}{12}$$

From which $P = 99.5$ lbs.

Hence, the mechanical advantage, $M = \frac{W}{P} = \frac{5000}{99.5} = 50.25.$

PROBLEMS

412. What is a machine? What is its utility? Define—Mechanical advantage, Velocity ratio, Efficiency, Reversed efficiency.

413. In a lifting machine an effort of 13 lbs. can lift a load of 1000 lbs., and an effort of 8 lbs. can lift 500 lbs. Determine the effort required to lift 700 lbs. If the mechanism is such that for a displacement of 1 foot of the load the effort undergoes a displacement of 80 feet, compute the efficiencies of the machine in all the cases. *Ans.* 10 lbs., 96.1%, 78.1%, 87.5%.

414. If the limiting load in a lifting machine is 1 ton and the velocity ratio is 100, find the mechanical advantage of the machine. The efficiency when the maximum load is lifted is 40%. By the application of an effort of 30 lbs. it is found that the efficiency of the machine becomes $\frac{5}{6}$ th. of the previous value. Determine the mechanical advantage in the second case. What should be the equation to represent the law of the machine? Determine the effort to lift a load of 1500 lbs. What is the efficiency at this load?

Ans. 40, 33.3, $P = .021\, W + 9$, 40.5 lbs., 37%

415. In a wheel-axle arrangement the axle is of square section, whose sides are 9 inches long. If the diameter of the wheel is 5 feet, determine the greatest and the least efforts to move a load of 320 lbs. slowly upwards. What is the velocity ratio in the arrangement?

Ans. 67.87 lbs., 48 lbs., 5.23.

416. In a wheel-axle arrangement to lift a load of 240 lbs. it is found that a pull of 4 feet raises the load by 5 inches only. What is the mechanical advantage of the machine? If the resistance in the machine is constant and absorbs 10% of the total work done, find the load that can be lifted by the effort. If the diameter of the wheel is 40 inches, what is the diameter of the axle? *Ans.* 9.6, 216., 4.166 inches.

417. In a differential wheel-axle arrangement the diameter of the wheel is 6 feet and the diameters of the axle are 12 inches and 6 inches respectively. Find the velocity ratio. If the load lifted is 480 lbs., determine the effort required to raise the load. *Ans.* 24, 20 lbs.

418. If the frictional resistances in the machine of the previous problem is taken into account and the efficiency of the machine is found to be 50%, determine the load lifted. *Ans.* 240 lbs.

419. The testing results in a wheel-axle arrangement is as follows:

When the load is 200 lbs., the effort is 10 lbs.

When the load is 360 lbs., the effort is 18 lbs.

The effort required to run the machine unloaded is 3 lbs.

Determine the law of the machine. If the resistance when the load is 400 lbs., is 15 lbs., what is the velocity ratio? *Ans.* $P = .05\,W + 3$, 50.

420. The pinion fixed on a driving shaft has 150 teeth. The driven wheel has 50 teeth. What is the speed of the driven wheel with respect to the driver? If the circular pitch of the pinions be 1.256 inches and if the horse-power with which the driver is working, be 5, determine the torque produced in the driven shaft, when the speed of the driver is 120 r.p.m. With what speed does the driven rotate? *Ans.* 875.7 lb. ins., 40 r.p.m.

421. In a winch (Fig. 323) the length of the effort handle is 15 inches. The number of teeth in the spur gear on the lever axle (T_1) is 45 and that of the gear fixed with the drum (T_2) is 360. If the diameter of the drum is 20 inches, determine what effort will be required to raise a load of 1000 lbs. when the machine is taken to be frictionless. If the efficiency of the machine is 40% what load can be raised by the same effort?

Ans. 83.3%, 400 lbs.

422. If a pull of 100 lbs. in the effort end of the chain in a differential pulley block lifts a load of 3000 lbs., determine the ratio between the two diameters in the upper sheave. Neglect friction. *Ans.* 15 : 14.

423. If in the previous problem the machine is 30% efficient, what should be the ratio? *Ans.* 9 : 7.

424. In the pulley systems, shown in Figs. 328 to 331, if the load is 1200 lbs. find the efforts taking the arrangements to be cent per cent efficient.

Ans. 600 lbs., 400 lbs., 100 lbs., 400 lbs.

425. If the efficiencies in the arrangements of the previous problem be equal and amount to 40%, what should be the efforts applied to raise the load by the different arrangements?

Ans. 840 lbs., 560 lbs., 140 lbs., 560 lbs.

426. If the arrangement of the pulley system shown in Fig. 330 is chosen for lifting purposes, and the load end (1) becomes the stationary support and the load is applied at the supported end (4), what is the velocity ratio of the machine? If the efficiency remains the same as in the previous problem, what is the effort required? *Ans.* 16, 105 lbs.

427. Is there any difference in the efforts applied by a man standing on the ground, if the direction of the pull be (1) vertical, and (2) at an angle of 30° with the vertical? Give reasons for your answer.

428. The diameters of the upper sheave in a differential pulley block are 13 and 11 inches respectively. The weight of the lower block is 3% of the net load that can be lifted by applying a pull of 200 lbs. at the effort end. Compute the amount of the net load, assuming that the machine is 30% efficient. *Ans.* 757.3 lbs.

429. In a combination of two worm and wheel arrangements for lifting purposes, the wheel of the second set is co-axially fitted on the same shaft of the worm of the first set. The drum, fixed with the wheel of the first set, has a diameter of 20 inches and the effort arm, attached to the worm of the second set, has a length of 15 inches. The worm of the first set is double-threaded and that of the second set is triple-threaded. The numbers of teeth in the wheels of the first and second sets are 80 and 45 respectively. What is the mechanical advantage of the machine when the effort is 20 lbs.? The efficiency in each set is 40 per cent. *Ans.* 144.

430. In the rod (Fig. 335) two nuts are introduced from two ends and the distance between them is made to become 3 inches. If the nuts are constrained to move in a direction along the axis of the rod only and if the pitches in both the portions are the same, will the distance change when the rod is rotated? If the right-handed has a pitch of $\frac{1}{8}$ inch and the left-handed $\frac{1}{16}$ inch, for how many rotations of the rod will the distance increase by $\frac{3}{4}$ inch? What is the direction of rotation? If the rod is rotated in the opposite direction by the same number of turns, what will be the distance?

Ans. Looking from the right hand if the rod is rotated clockwise the distance will decrease by $2p$ — the distance will increase by the same amount when rotated anti-clockwise, 4, anti-clockwise rotations looking from the right hand.
Will decrease by the same amount.

431. If in the previous problem the right nut is fixed and a load is attached to the left nut and the pitches are $\frac{1}{8}$ and $\frac{3}{16}$ inch in the right and the left portio respectively, determine the amount of load that can be drawn by an effort of 15 lbs. applied at the end of a 10-in. handle fixed at the right end. Assume the action of the frictional resistance as equivalent to half the load action. *Ans.* 2010 lbs.

432. If both the portions of the rod (Fig. 335) have right-handed screw of pitches, $\frac{3}{16}$ and $\frac{1}{16}$ inch respectively, what is the effect on the distance between the two nuts, for 8 turns right-handed looking from the right end? What is the effect for opposite turn?

Ans. 1 inch decrease, 1 inch increase.

433. A tie-rod for fastening two railway wagons together has two portions —one is a right-handed double-threaded screw and the other is left-handed double-threaded. The pitch is $\frac{1}{4}$ inch. The lever arm, 20 inches long, is attached to the rod at the middle. If the wagons with the load weigh 30 tons each and if the equivalent coefficient of friction for both rolling and sliding is .15, determine the effort required to draw the two wagons towards each other. *Ans.* 40 lbs.

434. In a pressing machine with differential screw mechanism (Fig. 337) the pressing block is attached with the nut N, and the effort is applied with the help of a hand-wheel of diameter 10 inches. If the pitches in the differential screw are $\frac{1}{8}$ (p_1) inch and $\frac{3}{16}$ (p_2) inch, both of them being right-handed, determine the average pressure exerted on the job by one rotation of the wheel with a force uniformly varying from zero at the beginning to 20 lbs. at the end. *Ans.* 5024 lbs.

435. In a jack-screw used for fitting and other overhauling purposes differential screw mechanism is adopted. The pitches in the screw are $\frac{1}{4}$ inch and $\frac{3}{16}$ inch respectively. If the machine is designed to lift a load of 20 tons and if the efficiency at this load is 40%, determine the effort required to work with the machine with the maximum load on it—the lever arm is 20 inches long. *Ans.* 22.24 lbs.

ADDITIONAL CHAPTER

BALANCING OF ROTATING BODIES

355. In case of rotating bodies if both ΣH and ΣV be equal to zero (Art. 275), the reaction at the support will be the same as when the body is at rest. But if they be not equal to zero, singly or both, the reaction at the support will be changed, that is, additional reactions will be produced at the support due to radial and normal components of the effective force in the opposite direction. These reactions always change in magnitude and direction with the motion and the position of the body with respect to the rotational axis. These additional reactions are called *Kinetic Reactions*, and the quantity which remains constant always, whether the body is at rest or in motion, is called the *Static Reaction*.

356. The ever-changing magnitudes and directions of the kinetic reactions are injurious to a part of a machine. The effect of the reactions is to set the part in vibration and to produce undue wearing in the bearing metal. Therefore, attempts are always made to eliminate this injurious effect. The method by which this can be done is called *Balancing*. This method is nothing but addition or removal, generally addition, of a quantity of mass at a definite point in the part so that the kinetic reactions due to the rotation of this quantity counter-balance the kinetic reactions of the part at the support, eliminating the fluctuation in the magnitudes and directions of the reactions, and thereby, the vibration and the undue wearing.

357. A body has two kinds of balances—statical, *i.e.*, balance when the body is at rest, and dynamical, *i.e.*, balance when the body is in motion. If a body has no tendency of rotation in any position whatsoever, when no effective force is applied, the body is said to be statically balanced. When the body rotates but there is no kinetic reaction at the support, the body is said to be dynamically balanced.

358. **Balancing in one Plane.** When a single rotating body of a machine, *e.g.*, the fly-wheel of an engine, is considered, it is found that if the part is statically balanced, then, automatically the dynamic balance of the part is maintained. It is clear that the C.G. of the mass is at the axis of rotation. If the C.G. is away from the axis

of rotation, then, the part is both statically as well as dynamically unbalanced. The part is balanced by adding a mass diametrically opposite to the C.G. on the other side of the axis, such that,

$W\bar{x} = w\bar{r}$, where W and w are the weights of the part and the added mass respectively and $\bar{x}$ and $\bar{r}$ are the distances of the centroids of the masses respectively from the axis of rotation.

When W rotates, the kinetic reactions due to both radial and normal or tangential components of the effective force, will be equal in magnitudes but opposite in direction to the reactions produced by the added mass, and the body is statically as well as dynamically balanced. The two centroids are on the same plane of rotation, which is at right angles to the axis of rotation and is simply named as *Transverse Plane*.

359. Balance in different Planes. Take the case of a shaft on which there are two identical similar parts as shown in Fig. 339 attached to the axle just on the opposite sides, so that the system is statically balanced. Now, if the system rotates the normal reactions of the two parts on the axis will form a couple, which will produce its effect on the bearings, and the radial components of the effective forces will also form a couple, the effect of which will also fall on the bearings. The two couples are at right angles to each other and therefore, cannot neutralise each other, though they may be equal in magnitudes. Again, a single force cannot balance a couple. Hence, in such cases two different masses are added in two different transverse planes in such a way that an opposite couple is induced in the plane of the resultant moment of the two previous moments so that it can neutralise the previous effects. The method is as follows: Let the C.G. of an unbalanced part whose weight is W be at a distance of $\bar{x}$ from axis of rotation (Fig. 340). Then, the body can be balanced by adding two masses whose weights are W_a and W_b respectively diametrically opposite to W on the other side of the axis in two transverse planes A and B.

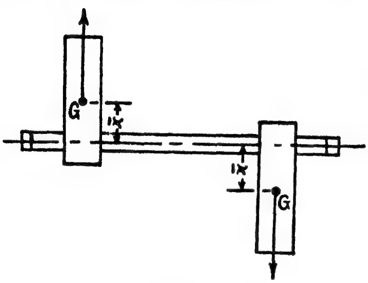


FIG. 339

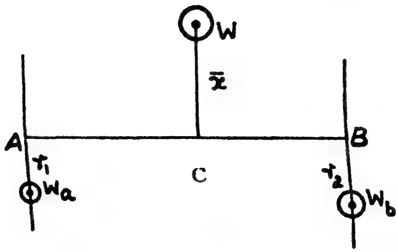


FIG. 340

The kinetic radial component of the force due to the motion of W s $\frac{W}{g}\omega^2 \bar{x}$ and the components due to the motion of the added masses ire $\frac{W_a}{g}\omega^2 r_1$ and $\frac{W_b}{g}\omega^2 r_2$ respectively. These three forces must be in equilibrium to maintain the balanced system. Therefore, taking moments about A and B respectively,

$$M_A = \frac{W}{g}\omega^2 \bar{x} \,.\, AC - \frac{W_b}{g}\omega^2 r_2 . AB = 0$$

$$M_B = \frac{W}{g}\omega^2 x \,.\, BC - \frac{W_a}{g}\omega^2 r_1 . AB = 0$$

From the two equations above,

$$W_b = \frac{\bar{x} . AC}{r_2 . AB} W, \text{ and } W_a = \frac{\bar{x} . BC}{r_1 . AB} W$$

'f r_1 and r_2 are assumed, then, the weights are easily determined. It s clear that if the radial component is neutralised, then, the tangential omponent also is automatically neutralised.

In the equations formed above it is clear that the quantity $\frac{\omega^2}{g}$, eing a common and constant factor, can be easily set aside from all he terms in computations. In the moment equations of the forces, hen, it is quite sufficient to retain the products of the weights of the masses and the distances of their centroids for computations. The products are named Wr products (pronouncing with the pronunciation f the alphabets).

Now, in case where there are more than one unbalanced mass in the same transverse plane, two masses are added to the two transverse planes for each of the attached masses. The method of determining the quantities is given above. The algebraic addition of the quantities in the two planes will give the weights of the masses added, when the attached masses are on the same longitudinal plane. But, in case where the attached masses are in different longitudinal planes the method is as follows :

Let the unbalanced system of weights be placed as shown in Fig. 341.

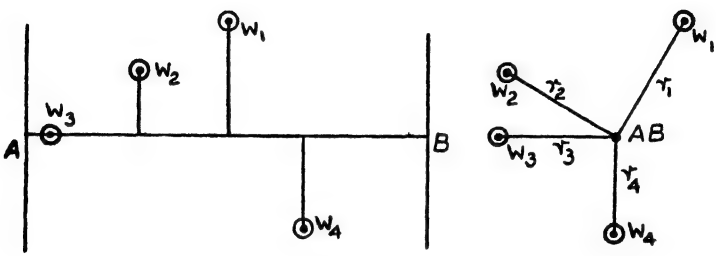


FIG. 341

First, the *Wr* products are determined for all the weights. Each of the *Wr* products is resolved into two components in two definite directions at right angles to each other, preferably horizontal and vertical. Then, by the principle of moments, the resultant horizontal and vertical *Wr* products on the two transverse planes are calculated. Next, compounding the two sets of components in each of the planes the final resultant *Wr* products with the directions are obtained. Now, dividing the products by an advantageous radial distance at which the added masses can be fixed, the weights of the added masses are determined. The method will be clear from the illustrated example 154.

Alternative Method — Dalvy's Method

This method is much easier than the previous one, because, (1) there is less chance of any mistake to be crept in, whereas, there is every possibility for this in a long computation with a number of weights, if the solution is not done with great patience and attention, and (2) the time required is much less in Dalvy's Method.

Introduce two forces, equal, opposite and parallel to $\frac{W}{g}\omega^2\bar{x}$ rejecting the common factor $\frac{\omega^2}{g}$, either at the plane A or at the plane B (say, at the plane A—Fig. 340). The system will remain unchanged. Then, in the plane A, there are a single force and a couple—one of the forces of the couple acting in the plane A opposite to the single force. The resultant of this single force and the couple is the single force $\frac{W}{g}\omega^2 x$ at C. Now, the moment of the couple, $\frac{W}{g}\omega^2\bar{x}$. AC is transformed into an equivalent couple whose arm is AB. Then, there will be two forces acting at A and one force at B. The two forces in the plane A are opposite in directions and their magnitudes are $\frac{W}{g}\omega^2\bar{x}$ in the upward direction and $\frac{W}{g}\omega^2\bar{x}\cdot\frac{AC}{AB}$ in the downward direction. The last one is also the force that is acting in the plane B but in the upward direction. The resultant of the two forces in the plane A will be a single force, as they are concurrent. Hence, if two masses are added in the two planes A and B in such a way that the forces induced are $\frac{W}{g}\omega^2\bar{x}\left(1-\frac{AC}{AB}\right)$ and $\frac{W}{g}\omega^2\bar{x}\frac{AC}{AB}$ respectively in the opposite direction, *i.e.*, in the downward direction (direction of the equilibrant in plane A), then, the system will have dynamical balance. If the two added masses be W_a and W_b respectively and if the distances of their centroids be r_1 and r_2 respectively, then,

$$W_a r_1 = W\bar{x}\frac{BC}{AB} \quad\text{and}\quad W_b r_2 = W x\cdot\frac{AC}{AB}.$$

The results corroborate with the results obtained by the method discussed previously.

In case where the number of unbalanced masses are more than one and in different longitudinal planes making different angles with a reference plane, then, each of the forces is resolved into a single force and a couple (the common factor $\frac{\omega^2}{g}$ is rejected and only Wr products are considered) as explained above. The Wr products of the forces and the moments of the Wr products are vectorially added separately. From the results of the vector addition, the final Wr

products and the final Wr product moment are obtained. The weights of the masses are added and the directions are obtained in the way as explained in the Illustrated Example 155.

Illus. Ex. 154. *In the four parallel planes of revolution A, B, C and D there is an unbalanced system of four masses of 20, 32, 24 and 40 pounds respectively attached to a shaft. The distances of their centres of gravity are 4, 3, 2 and 2.5 inches respectively. Their angular positions with respect to a plane containing the centroid of the first mass and the axis of the shaft are 0°, 90°, 150° and 240° respectively. The 2nd., 3rd. and the 4th. planes are at axial distances of 20, 40 and 50 inches respectively from the first plane (A). These four masses are balanced by two masses at radial distances of 4 inches in two planes P and Q, parallel to the plane of revolution at axial distances of 10 and 45 inches respectively from the plane A. Find the two balancing masses and their angular positions with respect to the previous reference plane.*

The weights of the masses with their radial distances from the axis of rotation and their angular positions are shown in the diagram (Fig. 342-I, End view).

First, find out the Wr products of the masses as shown in the diagram (Fig. 342-III, End view). Draw two views —Front (showing the vertical components of the radial distances—Fig. 342-II), and Top (showing the horizontal components—Fig. 342-IV). It is not necessary that the diagrams are to be drawn to scale, it is only required for references. Let P and Q be the two planes.

From the diagram IV, considering the horizontal components of the Wr products and taking moments about P, $\Sigma M_P = 0$, we get (taking clockwise moments positive),

$$Wr_Q \times 35 = 80 \times 10 + 48 \cos 30 \times 30 + 100 \sin 30 \times 40$$

$$\text{or,} \quad Wr_Q = \frac{800 + 1247 + 2000}{35} = 115.6$$

Again, taking moments about Q, $\Sigma M_Q = 0$, that is,

$$Wr_P \times 35 = 80 \times 45 - 48 \cos 30 \times 5 + 100 \sin 30 \times 5$$

$$\text{or,} \quad Wr_P = \frac{3600 - 208 + 250}{35} = 104.06$$

From diagram II, considering the vertical components of the Wr products, and taking moments about P, $\Sigma M_P = 0$, or,

$$Wr_Q \times 35 = 96 \times 10 + 48 \sin 30 \times 30 - 100 \cos 30 \times 40$$

$$\text{or,} \quad Wr_Q = \frac{960 + 720 - 3464}{35} = -51$$

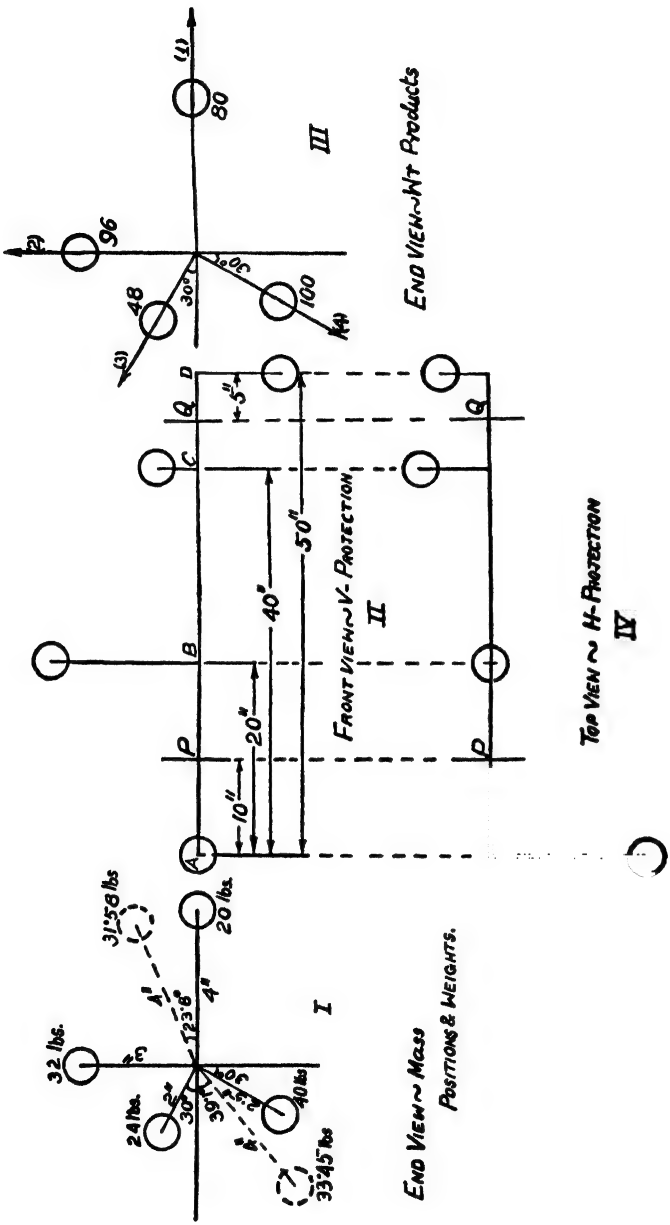

End View ~ Wt Products
III
Front View ~ V- Projection
II
Top View ~ H-Projection
IV
End View ~ Mass
Positions & Weights.
I
80
96
48
100
32 lbs.
24 lbs.
40 lbs
20 lbs.
31.58 lbs
33.45 lbs
10"
20"
40"
50"
5"

FIG. 342

Again, taking moments about Q, $\Sigma M_Q = 0$, or,

$$W r_P \times 35 = -96 \times 25 - 48 \sin 30 \times 5 - 100 \cos 30 \times 5$$

$$\text{or,} \quad W r_P = -\frac{2400 + 120 + 433}{35} = -84.37$$

Now, the resultant of $W r_Q$ (considering the H and V components)

$$= \sqrt{115.6^2 + 51^2} = 126.5$$

$$\theta = \tan^{-1} \frac{51}{115.6} = 23.8^\circ$$

Considering the positive and negative values of the moments it is clear that the weight added will be in the first quadrant at an angle of 23.8° with the horizontal axis, *i.e.*, with 0°.

Similarly, the resultant of $W r_P$ components,

$$= \sqrt{104^2 + 84.37^2} = 133.9$$

$$\theta = \tan^{-1} \frac{84.37}{104} = 39.1^\circ$$

Considering the positive and negative values of the moments it is clear that the weight added will be in the third quadrant at an angle of $180 + 39.1 = 219.1^\circ$ with 0°.

The weights being fixed at radial distances of 4 inches,

$$W_P = \frac{133.9}{4} = 33.5 \text{ lbs.}$$

$$W_Q = \frac{126.5}{4} = 31.625 \text{ lbs.}$$

Illus. Ex. 155. *The same problem as in illustrated example 154 has been taken to compare the results obtained by two different methods.*

Dalvy's Method

1. Find out the moments of Wr products about P paying attention to the convention of measuring the linear distances (positive or negative) and draw an angular space diagram for the moments as shown in Fig. 343-A.

2. Draw the force polygon (with the values and directions of Wr products in Fig. 342-III) *abcde*—Fig. 343-B. Then, *ea* is the equilibrant for the unbalanced system of Wr products.

3. Draw the vector polygon for the moments, 01234 (Fig. 343-C), with the help of the diagram in Fig. 343-A, from which the equilibrant $\vec{40}$ and the resultant $\vec{04}$ are obtained.

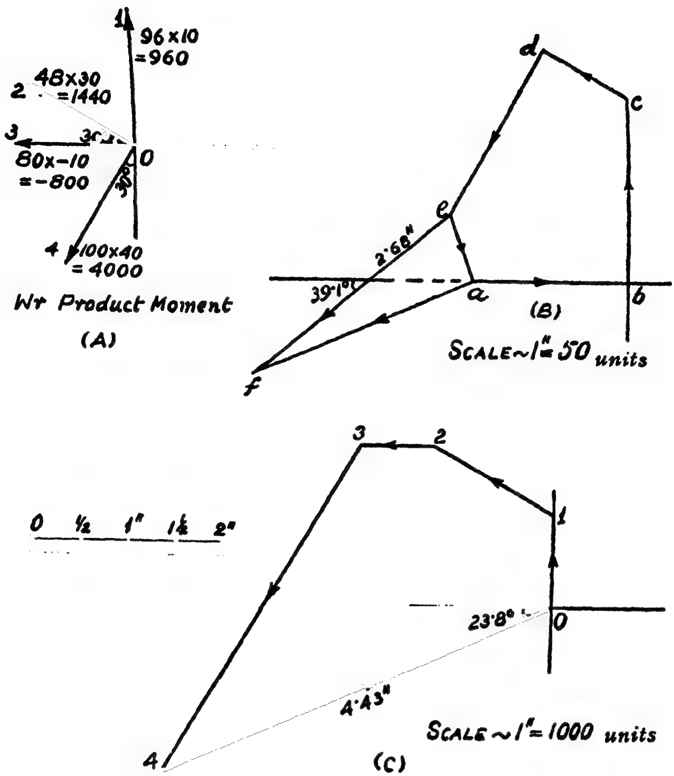


Fig. 343

The vector, $\overrightarrow{40} = 4.43$ inches and represents $4.43 \times 1000 = 4430$ moment alue.

Thus, the Wr product $= \dfrac{4430}{35} = 126.5$

This is the Wr product in the plane Q, which makes an angle of 23.8° ith the reference plane.

4. The Wr product obtained above will be acting just in the opposite rection in the plane P, in order to maintain equilibrium of the system. Now, lding this product vectorially (Fig. 343-B) the triangle of force *eaf* is obtain-l. The resultant is *ef* and measures 2.68 inches, which is equivalent to $.68 \times 50 = 134$, making an angle of $180 + 39.1 = 219.1°$ with the reference ·lane.

Thus, the weights of the masses added in the planes P and Q are, $\dfrac{134}{4}$ nd $\dfrac{126.5}{4}$ pounds weight, *i.e.*, 33.5 and 31.625 pounds weight respectively.

INDEX

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BOOKS OF REFERENCE

Applied Mechanics	Fuller and Johnston
Applied Mechanics	D. A. Low
Strength of Materials	Morley
Mechanics	Timoshenko and Young
Mechanism	Schwamb, Merrill and James

ERRATA

Page		Incorrect	Correct
84		position	piston
248	Prob. 229	The results of 2nd. and 3rd. will be interchanged	
253	Prob. 241 Third line	*BC*	*DC*
282		$T = \int_0^{t} \ldots\ldots$	$T = \int_0^{r_1} \ldots\ldots$
306	Eq. 148	$\bar{x} = \frac{\int dA\, y}{A}$	$\bar{x} = \frac{\int dA\, x}{A}$
		$\bar{y} = \frac{\int dA\, x}{A}$	$\bar{y} = \frac{\int dA\, y}{A}$
420		$+ \int_0^{\cdot 15} \ldots\ldots$	$+ \int_0^{\frac{1}{2}} \ldots\ldots$
479	Last but 2 lines	Sign of full stop	Sign of comma 'Thus' will be omitted
500	(Fig. 334-3)	flat bottom	rounded bottom